Robust Economic Implications of Nonlinear Pricing Kernels

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Background

- Hansen and Jagannathan (JPE, 1991) provided variance bounds for SDFs that price a set of returns.

- Useful for diagnosing asset pricing models, variance spanning tests, performance evaluation...
  
  - Linear projections of admissible SDFs on the payoff space of primitive assets.
  
  - Duality between mean-variance frontier for SDFs and mean-variance frontier for portfolios of returns.

- Why should one go beyond variance?

- Due to limitations in analyzing models and assets with nonlinear or non-Gaussian structure.
Objective

▶ Our goal: Better distinguish models, and more generally to robustify the analysis of models and asset’s performances.

▶ We provide a family of bounds each taking into account an infinite combination of moments in the space of SDFs.

▶ Based on the Cressie Read (1984) family of convex discrepancy functions.

▶ Easily captures higher moments.

▶ In contrast, HJ only consider minimizing the variance in the space of SDFs.

▶ Dual interpretation: each direction of SDF moments in the space of SDFs has equivalent in the space of returns.
Contribution and Related Literature

- How to account for higher moments?

- Our Cressie Read information bounds capture all the approaches above as particular cases.

- Most importantly, we show that the new bounds are relevant to improve robustness on diagnostic of some APMs and performance of some trading strategies.
Usefulness

- Nonlinear or non-Gaussian asset pricing models...
  - Bansal and Viswanathan (JF, 1993), Harvey and Siddique (JF, 2000), Dittmar (JF, 2002).
  - Disaster-based models (Barro (AER, 2006)), Long Run Risk models (Bansal and Yaron (JF, 2004)).

- Changing from HJ to our methodology matters to disaster models and LRR models.

- Performance of portfolios with options or other nonlinear assets (hedge funds).
  - Changing from HJ to our methodology changes performance of option strategies in more than 20%.
Methodology: Minimum Discrepancy SDF Bounds

▶ Let $R$ denote the vector of basis assets’ returns whose realizations $\{R_i\}_{i=1,\ldots,T}$ are given in a K-dimensional space.

▶ Assume a risk-free rate $R_f$.

▶ The moment conditions come from the Euler equations that any admissible SDF should satisfy to price the basis assets:

$$\frac{1}{T} \sum_{i=1}^{T} m_i (R_i - R_f) = 0_k \quad (1)$$

▶ The Minimum Discrepancy SDF bound is defined by:

$$\hat{m}_{MD} = \arg \min_{\{m_1, \ldots, m_T\}} \frac{1}{T} \sum_{i=1}^{T} \phi(m_i),$$

subject to $\frac{1}{T} \sum_{i=1}^{T} m_i (R_i - R_f) = 0, \quad \frac{1}{T} \sum_{i=1}^{T} m_i = \frac{1}{R_f}, m_i > 0 \forall i. \quad (2)$
Choosing the discrepancy: The Cressie Read (CR) Family

The CR discrepancy functions are given by:

\[
\phi(\pi) = \frac{(\pi)^{\gamma+1} - a^{\gamma+1}}{\gamma(\gamma + 1)}
\]  

(3)


It is simpler to solve the primal discrepancy problem in the dual space (Borwein and Lewis (1991)):

\[
\hat{\lambda} = \arg \sup_{\alpha \in \mathbb{R}, \lambda \in \Lambda} \frac{\alpha}{R_f} - \sum_{i=1}^{T} \frac{1}{T} \phi^{*,+} \left( \alpha + \lambda^T (R_i - R_f 1_K) \right),
\]  

(4)

where \( \Lambda \subseteq \mathbb{R}^K \) and \( \phi^{*,+} \) denotes the convex conjugate of \( \phi \) restricted to the positive real line:

\[
\phi^{*,+}(z) = \sup_{w > 0} zw - \phi(w)
\]  

(5)
The Implied SDF and the Information Frontier

- We can interpret CR dual problems as HARA utility maximizing problems:

\[
\hat{\lambda}_{CR} = \arg \sup_{\lambda \in \Lambda_{CR}} \frac{1}{T} \sum_{i=1}^{T} - \frac{1}{\gamma + 1} (1 + \gamma \lambda' (R_i - R_f))^{(\gamma+1)/\gamma}
\]  

(6)

- The implied SDF can be recovered via the first order conditions of the problem:

\[
\hat{m}_{MD}^i = \frac{T}{R_f} \frac{(1+\gamma \hat{\lambda}'_{CR}(R_i-R_f))^{1/\gamma}}{\sum_{j=1}^{T} (1+\gamma \hat{\lambda}'_{CR}(R_j-R_f))^{1/\gamma}}
\]

- The SDF-related frontier is found by solving (6) for a grid of meaningful values for the SDF mean \(A = \{a_1, a_2, ..., a_J\}\).

- The SDF-related frontier is given by:

\[
I_{CR}(a_l, \gamma) = \frac{1}{T} \sum_{i=1}^{T} \frac{\hat{m}_{MD}^i(a_l)^{\gamma+1} - 1}{\gamma(\gamma + 1)}, l = 1, 2, ..., J
\]  

(7)
Some Particular Estimators

- $\gamma = 1$, quadratic portfolio problem, implied SDF is a linear function of returns (HJ, 1991).

- $\gamma = 0$, CARA portfolio problem, implied SDF is an exponential function of returns, Stutzer (1995), Kitamura and Stutzer (1997, 2002).

- $\gamma = -1$, logarithmic portfolio problem, implied SDF is a hyperbolic function of returns, Bansal and Lehman (1997), Backus, Chernov and Martin (2012).

- $\gamma > 0$, captures the higher moment bounds proposed by Snow (1991).

- $\gamma < 0$, first appears in our contribution! Weights are given to combinations of even and odd higher moments of the SDF.
The Implied SDFs

- Cressie Read SDFs solving the bound problems will be **hyperbolic** functions of the basis assets returns $R$:

  \[
  \hat{m}_{MD}(R) = \beta \ast \left(1 + \gamma \hat{\lambda}_{opt}' (R - R_f)\right)^\frac{1}{\gamma}
  \]  \hspace{1cm} (8)

- They are **non-parametric** because only depend on the Lagrange Multipliers $\lambda_{opt}$ that solve the dual HARA portfolio problems.
  - Lagrange Multipliers come from Euler equations under the primal problem.

- They are **positive** because come from solutions to (strictly increasing) utility maximization problems.
Empirical Illustrations I: Diagnosing Asset Pricing Models

- We concentrate on two pervasive Consumption-based APMs: Disaster, and Long Run Risk (LRR).

- Both make important changes in the structure of the basic CCAPM, but very different in nature.

- Barro’s (2006) disaster model assumes that consumption growth (in log) comes from the sum of a normal and a jump component.

- LRR (Bansal and Yaron, 2004) assumes log-consumption growth is conditionally normal, but is highly persistent and has a persistent volatility.

- Power utility is kept in the disaster model, while an Epstein and Zin utility is adopted in the LRR model.
Taylor Expansion of the Cressie Read Function and Higher Moments

- The Cressie Read discrepancy is given by
  \[ E(\phi(m)) = \frac{E(m^{\gamma+1}) - a^{\gamma+1}}{(\gamma(\gamma+1))}. \]

- Taylor expanding \( \phi \) and taking expectations on both sides:
  \[ E(\phi(m)) = \frac{a^{\gamma-1}}{2} E(m - a)^2 + \frac{(\gamma - 1)a^{\gamma-2}}{3!} E(m - a)^3 + \frac{(\gamma - 1)(\gamma - 2)a^{\gamma-3}}{4!} E(m - a)^4 + \ldots \] (9)

- The region \( \gamma \leq 1 \): Investors like positive skewness in dual HARA problems.

- For \( \gamma < 1 \), all odd moments present negative weights and even moments present positive ones.

- For \( \gamma < -2 \), more weight to even moments than to odd moments.

- Negative region helps in separating models that generate more skewness from models that generate more kurtosis.
Skewness and Kurtosis Weights given by Cressie Read Estimators

How does the CR parameter $\gamma$ affects the weights given to skewness and kurtosis in solutions of our HARA-utility problems?

$$E[u(v)] \approx u(v0) + \frac{1}{2} u_2(v0) \lambda_{opt}^2 * E(R - E(R))^2 + \frac{1}{6} u_3(v0) \lambda_{opt}^3 * E(R - E(R))^3 + \frac{1}{24} u_4(v0) \lambda_{opt}^4 * E(R - E(R))^4$$

where $v0 = \lambda_{opt} * E(R - \frac{1}{a})$ is a scaled expected excess return.
The Disaster Model

- Consumption growth follows \( g_{t+1} = \eta_{t+1} + J_{t+1} \)

- where \( \eta_{t+1} \) is the normal component \( \mathcal{N}(\mu, \sigma^2) \) and \( J_{t+1} \) is a Poisson mixture of normals.

- Conditionally on the number of jumps, \( J_t \) is normal \( J_t | j \sim \mathcal{N}(j\alpha, j\lambda^2) \).

- The logarithm of the SDF with power utility is given by \( \log m_{t+1} = \log \beta - \zeta g_{t+1} \)

- This allows us to obtain analytical results for the Cressie Read discrepancies by taking expectations of \( \exp(\gamma + 1) \log m \).
Disaster Model

- We build two kinds of bounds: with market portfolio (S&P 500), and with option returns (including put options).

- Our model calibration follows Barro (2006) and BCM (2012): \( \tau = 0.01, \alpha = -0.3, \lambda = 0.15, \zeta = 6.8 \ldots \) keeping mean consumption growth equal to 0.02 and variance equal to 0.035\(^2\) (U.S. values).

- In our two illustrations we play with mean jump size \( \alpha \) and with risk aversion \( \zeta \).

- Example 1: Vary mean jump size \( \alpha \) from -0.3 to -0.1 and analyze model sensitivity.

- Example 2: Vary risk-aversion coefficient \( \zeta \) from 5 to 11.
It is easy for the disaster model to explain the equity premium puzzle, even with smaller values for the average jump size than the one considered in Barro (2006).
Diagnosing the Disaster Model with Entropic Bounds
Risk-Aversion in the Disaster Model with Entropic Bounds
The dynamics of the state variables in this model are:

\[
\begin{align*}
g_{t+1} &= \mu + x_t + \sigma_t \eta_{t+1} \\
x_{t+1} &= \rho x_t + \varphi \sigma_t e_{t+1} \\
\sigma_{t+1}^2 &= \sigma^2 + \nu_1 (\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1}
\end{align*}
\]

where \( g_t \) is the logarithm of real consumption growth. All innovations are \( \mathcal{N}, i.i.d. (0, 1) \).

The logarithm of the SDF is:

\[
m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{a,t+1}
\]

where \( r_{a,t+1} \) is the return on the wealth portfolio.

Bansal and Yaron (2004) use approximation techniques to obtain \( r_{a,t+1} \) and solve the model.

Important in our context: Persistences \( \rho \) and \( \nu_1 \) affect all moments of the log-normal SDF.
We calibrate the parameters following Bansal and Yaron(2004), but vary the persistence $\rho$ from 0.983 to 0.995.

We estimate bounds with two kinds of primitive assets: 1) Market return; 2) 6 Fama and French portfolios.

We estimate four entropic frontiers for $\gamma = -2, -1, 0, 1$.

In the paper we show that changing persistence $\rho$ affects all SDF moments but more kurtosis than skewness...
Consumption Growth Persistence in the LRR model with Entropic Bounds
Conclusion

- We provide bounds based on minimum discrepancy measures in the class of Cressie Read (1984).

- By duality, they can be obtained as solutions of portfolio problems with HARA utility functions.

- As a by product, we obtain non-parametric, non-linear positive admissible SDFs useful in performance analysis.

- We showed that bounds with large negative negative $\gamma$‘s are very useful to distinguish asset pricing models.

- In a related paper, we adopt Cressie Read MD estimators to generalize the HJ distance to diagnose APMs, and derive asymptotic properties of the estimators.