

# Appendix of the Paper: Identifying Volatility Risk Premia from Fixed Income Asian Options<sup>☆</sup>

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## Abstract

This Appendix is intended to complement the published version of the paper in the Journal of Banking and Finance. It is divided in two parts. In Appendix A, we provide proofs for the lemmas used to price Asian option under the proposed USV model. Appendix B presents expressions for bond conditional variances that are useful to obtain the volatility risk premium within the proposed model.

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## Appendix A - Proofs of Lemmas

In this section, we reproduce the model and the original lemmas provided in the article to keep the Appendix self-explanatory. The model is represented by the following Stochastic Differential System:

$$r_t = \phi_0 + X_t + Y_t + Z_t, \quad (1)$$

with:

$$dX_t = \kappa(\theta_t - X_t)dt + \sqrt{v_t}dW_X^{\mathbb{Q}}(t), \quad (2)$$

$$d \begin{bmatrix} Y_t \\ Z_t \end{bmatrix} = - \overbrace{\begin{bmatrix} \eta_Y & 0 \\ 0 & \eta_Z \end{bmatrix}}^{\eta} \begin{bmatrix} Y_t \\ Z_t \end{bmatrix} dt + \overbrace{\begin{bmatrix} \sigma_Y & 0 \\ \rho_{YZ} & \sigma_Z \end{bmatrix}}^{\sigma} \begin{bmatrix} dW_Y^{\mathbb{Q}}(t) \\ dW_Z^{\mathbb{Q}}(t) \end{bmatrix} \quad (3)$$

$$d\theta_t = \left(\gamma - 2\kappa\theta_t + \frac{v_t}{\kappa}\right)dt \quad \text{and} \quad (4)$$

$$dv_t = (\alpha - \beta v_t)dt + \delta\sqrt{v_t}dW_v^{\mathbb{Q}}(t); \quad (5)$$

where  $W_X^{\mathbb{Q}}$ ,  $W_Y^{\mathbb{Q}}$ ,  $W_Z^{\mathbb{Q}}$  and  $W_v^{\mathbb{Q}}$  are independent Brownian motions, and the volatility  $v_t$  follows a CIR (Feller) process<sup>1</sup>.

**Lemma 1.** *Let  $H(t, T) = \int_t^T r_u du$ , where the  $r_t$  dynamics appears in Equations (1)-(5). Then conditional on  $\mathcal{G}_{t, T}$ ,  $H(t, T)$  is normally distributed with mean  $M(t, T)$  and variance  $V(t, T)$  given by:*

$$M(t, T) = \phi_0\tau + M_X(t, T) + M_{YZ}(t, T)$$

and

$$V(t, T) = V_X(t, T) + V_{YZ}(t, T),$$

with

$$M_X(t, T) = -B_X(\tau)X_t + \int_t^T (1 - e^{-\kappa(T-u)}) \theta_u du, \quad (6)$$

$$V_X(t, T) = \frac{1}{\kappa^2} \int_t^T (1 - e^{-\kappa(T-u)})^2 v_u du, \quad (7)$$

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<sup>1</sup>A Feller process should satisfy the condition  $2\alpha > \delta^2$  in order to guarantee its positivity. See Cox et al. (1985) for a univariate application in interest rates, and Heston (1993) for an example of a bi-dimensional log-affine process with a Feller process driving stochastic volatility.

$$M_{YZ}(t, T) = -B_Y(\tau)Y_t - B_Z(\tau)Z_t \quad \text{and} \quad (8)$$

$$V_{YZ}(t, T) = 2A_{YZ}(t, T), \quad (9)$$

where  $A_{YZ}(t, T)$  is given by:

$$A_{YZ}(t, T) = \frac{\sigma_Y^2}{2\eta_Y} \left( \frac{-B_Y(\tau)^2}{2} + \frac{B_Y(\tau)+\tau}{\eta_Y} \right) + \frac{\rho_{YZ}^2 + \sigma_Z^2}{2\eta_Z} \left( \frac{-B_Z(\tau)^2}{2} + \frac{B_Z(\tau)+\tau}{\eta_Z} \right), \quad (10)$$

$v_u : u \in [t, T]$  is the path of the volatility conditional on  $\mathcal{G}_{t,T}$  and

$$\theta_u = e^{-2\kappa(u-t)} \left( \theta_t + \int_t^u e^{-2\kappa(t-s)} \left( \gamma + \frac{v_s}{\kappa} \right) ds \right), \quad t \leq u \leq T. \quad (11)$$

### Proof of Lemma 1

By definition of  $r_t$  we have

$$H(t, T) = \phi_0\tau + \int_t^T X_u du + \int_t^T (Y_u + Z_u) du.$$

From Brigo and Mercurio (2001) we know that  $\int_t^T (Y_u + Z_u) du$  conditioned on  $\mathcal{F}_t$  is normal with mean and variance given by Equations 8 and 9 respectively<sup>2</sup>.

Conditioning on the volatility path or equivalently, making  $v_t$  a deterministic function of time, it is not hard to verify by Ito's rule that for each  $t < T$  the unique (strong) solution of (2) is

$$X_T = \Xi(T) \cdot \left[ X_t + \kappa \int_t^T \Xi^{-1}(s)\theta_s ds + \int_t^T \Xi^{-1}(s)\sqrt{v_s} ds \right],$$

where  $\Xi(x) = e^{-\kappa(x-t)}$ . Stochastic integration by parts implies that

$$\int_t^T X_u du = \int_t^T (T-u) dX_u + (T-t) X_t. \quad (12)$$

By definition of process  $X_t$ , the integral in the right-hand side can be written as<sup>3</sup>

$$\int_t^T (T-u) dX_u = \kappa \int_t^T (T-u) (\theta_u - X_u) du + \int_t^T (T-u) \sqrt{v_u} dW_X(u).$$

<sup>2</sup>We refer to Karatzas and Shreve (1991) for more details about solutions of stochastic differential equations.

<sup>3</sup>In this appendix we drop the superscript  $\mathbb{Q}$  and denote the Brownian motion  $W^{\mathbb{Q}}$  simply by  $W$ .

But

$$\begin{aligned}
& \int_t^T (T-u) X_u du = \\
& = \kappa \int_t^T (T-u) \Xi(u) \int_t^u \Xi^{-1}(s) \theta_s ds du + X_t \int_t^T (T-u) \Xi(u) du + \\
& + \int_t^T (T-u) \Xi(u) \int_t^u \Xi^{-1}(s) \sqrt{v_s} dW_X(s) du.
\end{aligned}$$

Calculating separately the last two integrals, we have

$$\int_t^T (T-u) \Xi(u) du = \frac{T-t}{\kappa} + \frac{e^{-\kappa(T-t)} - 1}{\kappa^2}$$

and, again by integration by parts,

$$\begin{aligned}
& \int_t^T (T-u) \Xi(u) \int_t^u \Xi^{-1}(s) \sqrt{v_s} dW_X(s) du = \\
& = \int_t^T \left( \int_t^u \Xi^{-1}(s) \sqrt{v_s} dW_X(s) \right) d_u \left( \int_t^u (T-s) \Xi(s) ds \right) = \\
& = \left( \int_t^T \Xi^{-1}(s) \sqrt{v_s} dW_X(s) \right) \left( \int_t^T (T-s) \Xi(s) ds \right) - \\
& - \int_t^T \left( \int_t^u (T-s) \Xi(s) ds \right) \Xi^{-1}(u) \sqrt{v_u} dW_X(u) = \\
& = \int_t^T \left( \int_u^T (T-s) \Xi(s) ds \right) \Xi^{-1}(u) \sqrt{v_u} dW_X(u)
\end{aligned}$$

Substituting the previous terms in Equation 12 and after some algebraic manipulation, we obtain

$$\begin{aligned} \int_t^T X_u du &= \kappa \int_t^T (T-u)\theta_u du - \kappa^2 I(t, T) + \frac{1 - e^{-\kappa(T-t)}}{\kappa} X_t + \\ &+ \int_t^T \left( T - u - \kappa \int_t^T (T-s)e^{-\kappa(s-u)} ds \right) \sqrt{v_u} dW_X(u), \end{aligned}$$

where  $I(t, T) = \int_t^T (T-u) \int_t^u e^{-\kappa(u-s)} \theta_s ds du$ . Then

$$M_X(t, T) = \kappa \int_t^T (T-u)\theta_u du - \kappa^2 I(t, T) + \frac{1 - e^{-\kappa(T-t)}}{\kappa} X_t, \quad (13)$$

Using Ito's isometry we have

$$V_X(t, T) = \frac{1}{\kappa^2} \int_t^T (1 - e^{-\kappa(T-u)})^2 v_u du.$$

Now, changing the order of integration in  $I(t, T)$  we obtain

$$I(t, T) = \int_t^T \frac{T-u}{\kappa} \theta_u du + \int_t^T \frac{e^{-\kappa(T-u)} - 1}{\kappa^2} \theta_u du.$$

Substituting this expression in Equation 13 we conclude the proof.  $\square$

**Lemma 2.** *The time  $t$  price of a zero coupon bond maturing a time  $T$  can be written as*

$$P(t, T) = e^{-\phi_0 \tau - M(t, T) + \frac{V(t, T)}{2}},$$

that is,

$$\int_t^T (1 - e^{-\kappa(T-u)}) \theta_u du = \frac{V_X(t, T)}{2} - \frac{A_X(\tau)}{2} + B_\theta(\tau) \theta_t. \quad (14)$$

**Proof of Lemma 2**

By Equation 11 we have

$$\begin{aligned}
& \int_t^T (1 - e^{-\kappa(T-u)}) \theta_u du = \\
& \int_t^T e^{-2\kappa(u-t)} \left( \theta_t + \int_t^u e^{-2\kappa(t-s)} \left( \gamma + \frac{v_s}{\kappa} \right) ds \right) (1 - e^{-\kappa(T-u)}) du = \\
& \theta_t e^{2\kappa t} \int_t^T e^{-2\kappa u} (1 - e^{-\kappa(T-u)}) du + \\
& \int_t^T \int_s^T e^{-2\kappa u} (1 - e^{-\kappa(T-u)}) e^{2\kappa s} \left( \gamma + \frac{v_s}{\kappa} \right) duds = \\
& \frac{\theta_t}{2\kappa} (1 - e^{-\kappa(T-t)})^2 + \frac{1}{2\kappa} \int_t^T \left( \gamma + \frac{v_u}{\kappa} \right) (1 - e^{-\kappa(T-u)})^2 du = \\
& B_\theta(\tau)\theta_t - A_X(t, T) + \frac{V_X(t, T)}{2},
\end{aligned}$$

which concludes the proof. □

**Lemma 3.** *The time  $t$  price of a call option on the IDI with time to maturity  $T$  and strike price  $K$  is*

$$c(t, T) = \mathbb{E}^{\mathbb{Q}} [f(IDI_t, K, t, T, V(t, T)) | \mathcal{F}_t], \quad (15)$$

where

$$\begin{aligned}
& f(IDI_t, K, t, T, V(t, T)) = \\
& IDI_t \Phi(d) - KP(t, T) \Phi\left(d - \sqrt{V(t, T)}\right), \\
& d = \frac{\log \frac{IDI_t}{K} - \log P(t, T) + V(t, T)/2}{\sqrt{V(t, T)}}, \quad (16)
\end{aligned}$$

and  $\Phi(\cdot)$  is the standard normal cumulative distribution function.

### Proof of Lemma 3

Using the law of iterated expectations we have

$$\begin{aligned} c(t, T) &= \mathbb{E}^{\mathbb{Q}} \left[ \max (IDI_t - Ke^{-H(t, T)}, 0) \mid \mathcal{F}_t \right] = \\ &= \mathbb{E}^{\mathbb{Q}} \left[ \mathbb{E}^{\mathbb{Q}} \left[ \max (IDI_t - Ke^{-H(t, T)}, 0) \mid \mathcal{G}_{t, T} \right] \mid \mathcal{F}_t \right]. \end{aligned}$$

Then, the proof consists of a simple calculation of the ordinary integral  $\mathbb{E}^{\mathbb{Q}} \left[ \max (IDI_t - Ke^{-H}, 0) \mid \mathcal{G}_{t, T} \right]$ .

$$\begin{aligned} c(t, T) &= \mathbb{E}^{\mathbb{Q}} \left[ \max (IDI_t - Ke^{-H}, 0) \mid \mathcal{G}_{t, T} \right] = \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi V(t, T)}} \max (IDI_t - Ke^{-h}, 0) e^{-\frac{(h-M(t, T))^2}{2V(t, T)}} dh = \\ &= \int_{\log(K/IDI_t)}^{\infty} \frac{1}{\sqrt{2\pi V(t, T)}} (IDI_t - Ke^{-h}) e^{-\frac{(h-M(t, T))^2}{2V(t, T)}} dh. \end{aligned}$$

Making the substitution  $z = \frac{h-M(t, T)}{\sqrt{V(t, T)}}$  we have:

$$\begin{aligned} c(t, T) &= \int_{-d}^{\infty} \frac{1}{\sqrt{2\pi}} \left( IDI_t - Ke^{-z\sqrt{V(t, T)}-M(t, T)} \right) e^{-\frac{1}{2}z^2} dz = \\ &= IDI_t \int_{-d}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz - K \int_{-d}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z\sqrt{V(t, T)}-M(t, T)-\frac{1}{2}z^2} dz = \\ &= IDI_t \Phi(d) - Ke^{-M(t, T)+\frac{V(t, T)}{2}} \int_{-d}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z+\sqrt{V(t, T)})^2} dz. \end{aligned}$$

where  $d$  is given by Equation 16. Making a new substitution  $w = z + \sqrt{V(t, T)}$  and using Lemma 2 results in Equation 15.  $\square$

### Lemma 4.

$$\mathbb{E}^{\mathbb{Q}} (V(t, T) \mid \mathcal{F}_t) = V_{YZ}(t, T) + \frac{v_t}{\kappa^2} c_1(t, T) + \frac{\alpha}{\beta \kappa^2} c_2(t, T),$$

where:

- $c_1(t, T) = \frac{1-e^{-\beta\tau}}{\beta} - 2\frac{e^{-\beta\tau}-e^{-\kappa\tau}}{\kappa-\beta} + \frac{e^{-\beta\tau}-e^{-2\kappa\tau}}{2\kappa-\beta}$  and
- $c_2(t, T) = \frac{1}{\kappa} \left( -\frac{3}{2} + 2e^{-\kappa\tau} - \frac{e^{-2\kappa\tau}}{2} \right) + \tau - c_1(t, T)$ .

- $V_{YZ}(t, T)$  is given by Equation (9).

#### Proof of Lemma 4

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}(V(t, T)|\mathcal{F}_t) &= \\ V_{YZ}(t, T) + \frac{1}{\kappa^2} \int_t^T (1 - e^{-\kappa(T-u)})^2 \mathbb{E}^{\mathbb{Q}}(v_u|\mathcal{F}_t) du &= \\ V_{YZ}(t, T) + \frac{1}{\kappa^2} \int_t^T (1 - e^{-\kappa(T-u)})^2 \left( v_t e^{-\beta(u-t)} + \frac{\alpha}{\beta} (1 - e^{-\beta(u-t)}) \right) du, \end{aligned}$$

where in last step we have used the property of the mean of a CIR process (see Brigo and Mercurio (2001)). Expanding the terms in the right side and calculating the ordinary integrals give the desired result.  $\square$

#### Appendix B - Conditional Variance in Affine Models

Bond yields conditional variances under general affine models were provided in Almeida et al. (2006), and Jacobs and Karoui (2006). We specialize their results to the model proposed here.

The covariance matrix  $var_t(E_{t+s})$  can be calculated by the following algorithm<sup>4</sup>:

1. Let  $D \in \mathbb{R}^{25 \times 5}$  be the matrix such that  $D_{ij} = 1$  if  $i = 5(j - 1) + j$  and 0 otherwise.
2. Consider the matrices

$$\begin{aligned} \kappa_e &= \begin{bmatrix} \kappa & 0 & 0 & -\kappa & 0 \\ 0 & \eta_y & 0 & 0 & 0 \\ 0 & 0 & \eta_z & 0 & 0 \\ 0 & 0 & 0 & 2\kappa & -1/\kappa \\ 0 & 0 & 0 & 0 & \beta \end{bmatrix}, \\ \theta_e &= \begin{bmatrix} \frac{\gamma + \frac{\alpha}{\beta\kappa}}{2\kappa} & 0 & 0 & \frac{\gamma + \frac{\alpha}{\beta\kappa}}{2\kappa} & \frac{\alpha}{\beta} \end{bmatrix}', \end{aligned}$$

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<sup>4</sup>For more details on this computation see Fackler (2000).

$$\Sigma_e = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_Y & 0 & 0 & 0 \\ 0 & \rho_{YZ} & \sigma_Z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\alpha_e = [0 \ 1 \ 1 \ 0 \ 0]'$$

and

$$\beta_e = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta^2 \end{bmatrix}.$$

3. Next, compute the  $30 \times 30$  matrix

$$K = \begin{bmatrix} \kappa_e & 0 \\ -(\Sigma \otimes \Sigma) D\beta_e & \kappa_e \otimes I_5 + I_5 \otimes \kappa_e \end{bmatrix}$$

and the vector  $\Theta \in \mathbb{R}^{30}$  defined by

$$\Theta = \begin{bmatrix} \kappa_e \theta_e \\ (\Sigma \otimes \Sigma) D\alpha_e \end{bmatrix},$$

where  $\otimes$  stands for the Kronecker product operator and  $I_n$  is the  $n \times n$  identity matrix.

4. For a fixed time to maturity  $s$ , compute the vector  $Q_0 \in \mathbb{R}^{30}$  and the matrix  $Q_1 \in \mathbb{R}^{30 \times 5}$  defined as

$$Q_0 = (I_{30} - e^{-Ks}) K^{-1} \Theta$$

and

$$Q_1 = e^{-Ks} \begin{bmatrix} I_5 \\ 0 \end{bmatrix}$$

5. Then

$$vec(var_t(E_{t+s})) = V_0 + V_1 E_t$$

where  $vec$  denotes the vectorized representation of a matrix,  $V_0 \in \mathbb{R}^{25}$  is the vector composed by the last 25 elements of  $Q_0$  and  $V_1 \in \mathbb{R}^{25 \times 5}$  is the matrix formed by the last 25 lines of  $Q_1$ .

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