A Generalization of Principal Component Analysis For Non-Observable Term Structures in Emerging Markets

Caio Ibsen Rodrigues de Almeida *1
Antonio Marcos Duarte, Jr. **
Cristiano Augusto Coelho Fernandes ***

Abstract

Principal Component Analysis (PCA) has been traditionally used for identifying the most important factors driving term structures of interest rates movements. Once one maps the term structure dynamics, it can be used in many applications. For instance, portfolio allocation, Asset/Liability models, and risk management, are some of its possible uses. This approach presents very simple implementation algorithm, whenever a time series of the term structure is disposable. Nevertheless, in markets where there is no database for discount bond yields available, this approach can't be applied. In this article, we exploit properties of an orthogonal decomposition of the term structure to sequentially estimate along time, term structures of interest rates in emerging markets. The methodology, named Legendre Dynamic Model (LDM), consists in building the dynamics of the term structure by using Legendre Polynomials to drive its movements. We propose applying LDM to obtain time series for term structures of interest rates and to study their behavior through the behavior of the Legendre Coefficients levels and first differences properly normalized (Legendre factors). Under the hypothesis of stationarity and serial independence of the Legendre factors, we show that there is asymptotic equivalence between LDM and PCA, concluding that LDM captures PCA as a particular case. As a numerical example, we apply our technique to Brazilian Brady and Global Bond Markets, briefly study the time series characteristics of their term structures, and identify the intensity of the most important basic movements of these term structures.

* IBMEC-RJ and Department of Mathematics, Stanford University, cibsen@math.stanford.edu
** IBMEC-RJ, Brazil, aduarte@ibmecrj.br
*** Pontifícia Universidade Católica do Rio de Janeiro, Brazil, cris@ele.puc-rio.br .
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Introduction

Principal Component Analysis (PCA; Mardia et al. [1992]) has been traditionally applied to identify the main factors responsible for driving the movements of term structures of interest rates (Litterman and Scheinkman [1991], Barber and Copper [1996], Singh [1997] and Rebonato [1997]). The only conditions required by PCA are stationarity and serial independence of the time series of interest rates increments. Whenever the term structure is observable[1], an easy application of PCA over the interest rates historical variations (for each term structure node) provides the above mentioned factors. Nevertheless, when the term structure is non-observable, which are the cases for emerging markets Eurobonds, Brady Bonds and any other market with predominantly coupon bonds, there is no possibility of a direct application of PCA. In this case, a procedure for the construction of the term structure time series appears as a necessary solution. It should begin with the adoption of a model for the evolution of the term structure.

Many works, on finite dimensional Markovian models for the term structure, have been proposed during the last decades. Some of them model the evolution of the short term rates (see for instance, Vasicek [1977], Cox et al. [1985]), while others are multi-factor models, based on the evolution of finite sets of state variables, usually yields or forward rates (Duffie and Kan [1996], El Karoui et al. [2000]). The empirical versions of these models are usually estimated by a direct application of Maximum Likelihood (Chen and Scott [1993], Pearson and Sun [1994]). However, whenever it is assumed that the whole term structure is observed under measurement errors, the likelihood is obtained with the use of auxiliary filtering techniques. One adopts a linear Kalman Filter (Pennachi [1991], Duan and Simonato [1995]) or non-linear one (Lund [1997], Duffee and Stanton [2001]), according to if the relation between the observed quantities and the state variables is linear or non-linear.

In this work, we propose a multi-factor model for the evolution of term structures in emerging markets, using a specific orthogonal basis of functions to parameterize the credit spread risk over a risk-free benchmark curve. This model, which we name Legendre Dynamic Model (LDM), evolves the term structure through a linear combination of state variables directly attached to the Legendre polynomials, combined with the own dynamics of the benchmark curve. It presents at least two

[1] For instance, the term structure of the U.S. strips or any other market composed of zero coupon instruments.

good features: First, it captures PCA as a particular case. Second, it is a specific choice on the class of the exponential affine factor models, fully described in Duffie and Kan [1996].[2]

Although we formulate the model in a general state space format, from the estimation viewpoint, we adopt a two-step procedure similar to the one proposed in Diebold and Li [2002]. First, we sequentially estimate the Legendre coefficients (levels) and Legendre factors (normalized first differences) along the time dimension, obtaining their time series. After that, we fit time series models to the coefficients and factors, and test some forecasts in out of sample data.

The article is organized as follows. Section 2 presents the LDM model, i.e., the basic equations for the term structure of interest rates evolution and the estimation process. Section 3 points out the relation between the LDM and the PCA models. Section 4 presents a numerical example applied to the Brazilian Sovereign Market. Finally, Section 5 concludes the article.

2. Legendre Dynamic Model

2.1 Term Structure of Interest Rates in Emerging Markets

According to Almeida et al. [1998], the term structure of interest rates in an emerging market can be modeled as:

$$R(t) = B(t) + \sum_{n \geq 0} c_n P_n \left(\frac{2t}{\ell} - 1\right), \ \forall t \in [0, \ell]. \tag{1}$$

where t denotes time to maturity, $B(t)$ denotes a benchmark curve, P_n is the Legendre polynomial of degree *n* (Lebedev [1972]), c_n is a parameter (named Legendre coefficient) and ℓ is the longest maturity of a bond in the emerging market under consideration.

In this context, obtaining the term structure consists on the problem of estimating the parameters $c_0, c_1, c_2, ...,$ in Equation (1).

We define the discount function as:

$$D(t) = e^{-R(t)t}, \forall t \in [0, \ell]. \tag{2}$$

[2] They present necessary and sufficient conditions that should be satisfied by the SDE that drive the state variables dynamics in order to guarantee an arbitrage-free model. Following a similar approach, Almeida [2002] shows that it is always possible to build an arbitrage-free LDM.

We assume that m bonds are available in a particular instant of time for the estimation process. The estimation of the coefficients $c_0, c_1, c_2, ...,$ is accomplished by the use of a non-linear regression equation given by:

$$p_j + a_j + 1_j^{put} o^p - 1_j^{call} o^c = \sum_{l=1}^{f_j} u_{jl} D(t_{jl}) + e_j, \forall j = 1,2,...,m, \tag{3}$$

where p_j denotes the price of the j^{th} bond, a_j denotes the accrued interest of the j^{th} bond, 1_j^{put} and 1_j^{call} are dummy variables indicating the existence of embedded put and call options in the bond, o^p and o^c are unknown parameters related to the prices of the embedded put and call options, f_j denotes the number of remaining cash flows of the j^{th} bond, and t_{jl} the time remaining for payment of the l^{th} cash flow u_{jl} of the j^{th} bond; e_j is the random disturbance, with $E(e_j) = 0$, $E(e_j^2) = \sigma_j^2$, $E(e_j, e_i) = 0 \; \forall j \neq i$.

2.2 Term Structure Evolution: The State Space Form

In this section, the cross-sectional problem described in Sub-Section 2.1 is transformed into a problem that considers simultaneously the cross sectional and time series implications for the term structure. The data consists of observations of the prices of the bonds sampled at instants $\tau_1, \tau_2, ...\tau_l$. The observations in a fixed time τ_k consist of a $m_k x1$ vector, where m_k denotes the number of bonds available at time τ_k. The model is based on a dynamic version of Equations (1), (2) and (3), considering that we fix a finite number of Legendre polynomials:

$$R(t,\tau) = B(t,\tau) + \sum_{n=0}^{N-1} c_n(\tau) P_n(\frac{2t}{\ell} - 1) \tag{4}$$

$$D(t,\tau) = e^{-R(t,\tau)t} \tag{5}$$

$$p_j(\tau) + a_j(\tau) + 1_j^{put}(\tau) o^p - 1_j^{call}(\tau) o^c = \sum_{l=1}^{f_j(\tau)} u_{jl} D(t_{jl}, \tau) + e_j(\tau), \forall j = 1,2,...,m_\tau \tag{6}$$

The Legendre coefficients $\{c_n\}_{n=0,...N-1}$, play the role of the state variables, and we restrict $\left\{ e_\tau = \begin{bmatrix} e_1(\tau) \\ \vdots \\ e_{m_\tau}(\tau) \end{bmatrix} \right\}$ to be independent over time. It is also worth mentioning that the linear combination of the Legendre polynomials in Equation (4) represents the credit spread function over the benchmark curve and will be denoted by $C(t,\tau)$.

Actually, we are interested in obtaining the evolution of the unobserved Legendre coefficients, which are non-linearly related to the observed prices of the bonds. The model is completed when we introduce an equation that parameterizes the dynamics of the state space variables:

$$X_k = g(X_{k-1}, \psi) + \varepsilon_k \tag{7}$$

where X_k denotes the column vector containing the coefficients $\{c_n(\tau)\}_{n=0,...N-1}$ at time τ_k, ψ is a vector of parameters, $g : \Re^N \to \Re^N$ is a general function, and $\{\varepsilon_\tau\}$ is Gaussian, independent over time, and also independent of $\{e_\tau\}$.

2.3 Estimation Process: Kalman Filter x Sequential Estimation

The model, in its most general form, supports a non-linear measurement equation combined to a non-linear function for the evolution of the state space variables. The estimation process should be accomplished by the use of a non-linear Kalman filter. We present a brief description of the filter and refer to Lund [1997] for a good treatment on it.

The filter goal is to estimate the unobserved state variables, and to estimate the parameter vector ψ. It is implemented in two steps: the prediction step and the estimation step. The prediction step, at instant τ_k, uses the information available on the observed prices up to instant τ_{k-1} to derive the conditional distribution of the state vector X_k. Then, the conditional expectation of the state vector X_k, on the available set of information up to instant τ_{k-1} is used to predict X_k. This is the best estimator in the Mean Square Sense. In the update step, additional information available on the prices of the bonds in instant τ_k is used to obtain a better estimation of X_k. The parameters are estimated by maximizing the log-likelihood function of the observed prices, which is obtained by recursive application of the two steps described above. It is a computer intensive procedure, which will not be adopted in the numerical example of this work.

Suppose in stead of adopting the Kalman filter approach, one decides to sequentially estimate the non-linear regression described by Equations (1), (2) and (3), considering the state variables $\{X_k\}$ as parameters and eliminating the parameters ψ. What kind of assumption would allow this estimation to be consistent? The answer is simple, though difficult to happen in the real world: the state variables should be independent over time, i. e., the function *g* should be the null function. Then, at instant τ_k, the distribution of the observed prices $p(k)$ is independent of the set of information cumulated up to instant τ_{k-1}. In this case, the values of the estimated state variables would be the ones that minimize the following function[3]:

$$\sum_{k \le l} \sum_{j=1}^{m_k} e_j(k)^2 \tag{8}$$

where $\{e(k)\}$ is the vector of innovations which appears in Equation (6).

Especially after the works of Nelson and Siegel [1987] and Svensson [1994], many practical works have relied on this kind of sequential estimation (see Alonso et. al [2000], Dahlquist et. al [2000], Anderson and Sleath [1999] and Clare and Lekkos [2000]). Despite its theoretical time-inconsistency (in the cases where there is temporal dependence between the state variables) and of frequently being criticized (see Bliss [1997] or Filipovic [1999]), it is still adopted. Why? It is easy to implement and it may give some insights about the behavior of the term structure dynamics by identifying the "non-parametric" temporal dependence that appears among the state variables. In other words, the results of this naïve estimation process may help to define parametric relations for the evolution of the state variables to be used in the Kalman filter. So, as a first simple study, in the numerical example of Section 4, we adopt this naïve approach to collect information about term structures in emerging markets, particularly the types and intensities of the main movements of these curves.

3. Legendre Factors and Principal Components

The application of PCA assumes two basic properties that increments of data must satisfy: stationarity and independence. It is easy to observe from Equation (7), that this independence requirement is rather restrictive, precisely because in the majority of choices for function *g*, the increments $\{\Delta X_\tau\}$ won't be independent. Nevertheless, take for instance, *g* as the identity function in order to obtain independent increments. Note in this case, that we are forcing the dynamics of the spread function to be driven by a set of finite state variables, attached to orthogonal functions of the maturity variable t, with independent and stationary increments. Intuitively, we should be able to extract principal components from the spread function $C(t,\tau)$. Moreover, we may expect that the Legendre factors, i.e., the normalized increments of the state variables, should be equivalent to the principal components obtained from the spread function data. That is exactly what we will show now.

Take a partition ξ formed by *M* maturity times $0 = t_1 < t_2 < ... < t_{M-1} < t_M = \ell$ uniformly spaced in the maturity interval $[0,\ell]$. Suppose we are observing the term structure of interest rates in these maturity instants. Define the Legendre factors by:

$$L_f(k) = \Gamma \Delta X_k \tag{9}$$

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[3] This minimization is equivalent to maximizing the log-likelihood function.

$$\Gamma = diag\left(\left\| P_0 \right\|_\xi \quad \cdots \quad \left\| P_{N-1} \right\|_\xi \right) \tag{10}$$

where Γ is a diagonal matrix and $\left\| P_n \right\|_\xi$ [4] is the Euclidean norm of the Legendre polynomial of degree *n*, sampled in the partition ξ:

$$\left\| P_n \right\|_\xi = \sqrt{\sum_{i=1}^{M} P_n^{\;2}\left(\frac{2t_i}{\ell} - 1\right)} \tag{11}$$

The dynamics of the sampled spread function (omitting the variable *t*) becomes:

$$\Delta C_\xi(\tau) = A_\xi \cdot L_f(\tau) \tag{12}$$

where A_ξ is a MxN matrix with each column containing the ordered and normalized Legendre polynomials sampled in ξ:

$$\widetilde{P}_n(t) = \frac{P_n\left(\frac{2t}{\ell} - 1\right)}{\left\| P_n \right\|_\xi}, t = t_1, t_2, ..., t_M \tag{13}$$

$$A_\xi = \begin{bmatrix} \widetilde{P}_0(t_1) & \widetilde{P}_1(t_1) & \cdots & \widetilde{P}_{N-1}(t_1) \\ \widetilde{P}_0(t_2) & \widetilde{P}_1(t_2) & \cdots & \widetilde{P}_{N-1}(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{P}_0(t_M) & \widetilde{P}_1(t_M) & \cdots & \widetilde{P}_{N-1}(t_M) \end{bmatrix} \tag{14}$$

Why is it possible to apply PCA to the increments of the sampled spread function? The answer is simple. Take an arbitrary entry of the increments of this function to be analyzed, for instance: $\Delta C_\xi(t_j, \tau) = \sum_{i=0}^{N-1} P_i(t_j) L_{f\,i}(\tau)$. Due to the independence and stationarity of the increments of the state variables $\{X_\tau\}$ over time, and to the fact that this entry is expressed as a linear combination of these increments, it is also stationary and independent over time. Now, if we apply PCA on $\Delta C_\xi(\tau)$ we obtain *M* orthogonal factors in the state space, which describe the same

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[4] While the norm of the polynomial in $C[-1,1]$ is $\left\| P_n \right\|_{C[-1,1]} = \sqrt{\int_{-1}^{1} P_n^{\;2}(t)dt} = \sqrt{\frac{2}{2n+1}}$

dynamics for $\Delta C_{\xi}(\tau)$ as the one described by the Legendre factors. Formally speaking, name the covariance matrix of $\Delta C_{\xi}(\tau)$ by $\mathrm{cov}(\Delta C_{\xi})$, where we suppress the time index under the stationarity assumption. Order its eigenvalues[5] in decreasing order of magnitude. The jth principal component is a linear combination of the sampled spread increments $fac_{\xi j} = \varphi_j^{T} \Delta C_{\xi}$, where the weights come from the eigenvector of the matrix $\mathrm{cov}(\Delta C_{\xi})$ correspondent to the jth eigenvalue (Campbell [1997])[6].

It is not difficult to prove that the following two limits are valid[7]:

$$\lim_{\|\xi\|\downarrow 0} \|\xi\| \|P_i\|_{\xi}^{2} = \|P_i\|_{C[-1,1]}^{2} \tag{15}$$

$$\lim_{\|\xi\|\downarrow 0} < \widetilde{P}_i, \widetilde{P}_j > = \int_{-1}^{1} P_i(t) P_j(t) dt = \begin{cases} 1, \text{se } i = j \\ 0, \text{se } i \neq j \end{cases} \tag{16}$$

These equations show that when we make the norm of the partition converge to zero (by taking thinner grids), the sampled Legendre polynomials converge, "in some sense", to the original Legendre polynomials embedded in the space $C[-1,1]$. This fact is crucial to allow us to approximate, under Frobenious norm (Golub and Van Loan [1985]), the matrix A_{ξ} in Equation (14) by a matrix with orthogonal columns A. This approximation forms the basis for the results obtained in this section, because the orthogonality of the columns of A allows us to directly relate the eigenvectors of the spread function to the eigenvectors of the Legendre factors.

Now let's compare the Legendre factors and the principal components. There are two main differences between them. First, while the number of principal components depends on the number of points in the partition, for any partition, there are only N Legendre factors. Second, by construction, the principal components are orthogonal while we can't guarantee this feature for the

[5] Almeida [2001] presents the necessary conditions for the existence of a unique spectral decomposition of matrix $\mathrm{cov}(\Delta C_{\xi})$.

[6] Some authors define the principal components as the eigenvectors, in stead of the linear combination that they produce (James and Webber [2000]).

[7] The proofs are also provided in Almeida [2001].

Legendre factors. So, why do we claim that these two sets of factors are equivalent? Because both generate the same dynamics for the spread function, and both are asymptotically attached to sets of orthogonal vectors. Almeida [2001] presents the proofs of the following two simple lemmas that are sufficient to show the equivalence of Legendre factors and principal components:

Lemma 1. Only *N* eigenvectors of $\text{cov}(\Delta C_{\xi})$ are related to non-null eigenvalues.

In other words, only *N* principal components generate non-negligible movements.

Now, let G_{ξ} denote the matrix, which contains the eigenvectors of $\text{cov}(L_f)$, the covariance matrix of the Legendre factors.

Lemma 2. The Legendre factors are related to the non-negligible principal components through the following orthogonal transformation:

$$L_f(\tau) = G_{\xi} fac_{\xi}(\tau) \tag{17}$$

Moreover, the matrix Ψ containing the eigenvectors of $\text{cov}(\Delta C_{\xi})$ may be approximated by:

$$\Psi = A_{\xi} G_{\xi} \tag{18}$$

In particular, when this orthogonal transformation G_{ξ} is the identity, the Legendre factors and the principal components are coincident. In fact, we could not expect anything different than that. We use an orthogonal basis of functions to simultaneously build and drive the movements of the term structure. The worse that could happen would be to obtain the increments of the state space variables, the Legendre factors, not orthogonal. In this case, Lemma 2 proposes the application of a simple orthogonal transformation to the Legendre factors and to the Legendre polynomials, which generates the so wanted orthogonal factors, and their correspondent eigenvectors.

4. Identifying the Main Types of Movements of Term Structures in Emerging Markets: A Numerical Example

We rely on sequential estimation to construct a database for the emerging markets term structures. In order to estimate the term structures in a specific instant, we apply a similar version of the model presented in section 2.1 (Almeida et. al [2000]), to jointly estimate the term structures of Brazilian Brady Bonds and Global Bonds Markets[8]. The term structures were estimated for the period between 10/13/99 and 11/10/2000. Figure 1 presents these curves. Three Legendre polynomials were used to generate all possible movements of these term structures, hence generating a three-factor model: the Legendre polynomial of degree zero (translation), the Legendre polynomial of degree one (rotation), and the Legendre polynomial of degree two (torsion).

Figure 2 presents charts with the time series of the Legendre coefficients (state variables) for both markets[9]. These are very informative about the term structure behavior. Take for instance the translation Legendre coefficient. It captures the level of risk over the benchmark, perceived by the investors along time. For the Global market, it presents a mean of 672 basis points, and attained its minimum value in the beginning of year 2000. The rotation Legendre coefficient, on its turn, represents approximately half of the spread between short and long rates (see the Legendre polynomials in Figure 7). It is useful to measure the risk premium in the term structure. In our model it represents risk premium for both the Global bonds and Brady bonds markets. Note the in the beginning of year 2000 it also achieves its minimum, confirming that it was a calm period with good long term expectations from the investors point of view. The torsion Legendre coefficient measures the degree of curvature of the term structure, and it is very sensitive to changes in

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[8] see Almeida et. al [2001] for an application in risk management and Fabozzi [1997] for definitions of these markets.

[9] The adopted model says that the Global Bonds term structure can be obtained from the U.S. strips term structure through a translation, a rotation and a torsion, while the Brady Bonds term structure can be obtained from the Global Bonds term structure through a translation, and a torsion. This means that the rotation Legendre coefficient is common to the Brady and Global curves.

investor expectations related to medium term events. When it is negative, it generates a concave term structure, when positive it generates a convex term structure. Values close to zero indicate less uncertainty on investors' expectations. High positive or negative values usually come combined with high values for the level of the term structure (translation coefficient). This coefficient deserves a separate analysis for each market. Note that the torsion coefficient related to the Global market is negative for the whole period; it is oscillating around 250 basis points, and achieves its minimum (in absolute value) also in the beginning of year 2000. On the other hand, the one related to the Brady market has a descendent trajectory, which begins with high positive values (around 200 basis points) and finishes with high negative values in November 2000. Note that it goes close to zero in the beginning of year 2000, when the translation coefficient of the Brady market also achieves a very low value, increases again, together with the translation coefficient and then decreases to a value close to zero in the end of February, together with the translation coefficient. So, it is clear with this small analysis that the time series of the coefficients may bring good insights about the behavior of the fixed income markets analyzed using the model.

The next step was to adjust linear models to the coefficients series in order to verify its potential to forecast movements in the term structures. We adjusted both ARMA models (Box et. al [1994]) and a Vector Autoregressive Model (VAR; Hamilton [1994]) to these variables and found out that in general, they present auto-regressive behavior with roots near unity, similar characteristics to the ones reported in Backus et. al [1999] for the U.S term structure.

Figure 3 presents charts with the time series of the Legendre factors for both markets. We applied the BDS test (Brock et. al [1991]) and the Unit Root ADF Test (Davidson and MacKinnon [1993]) to verify, respectively, the hypothesis of independence and stationarity of the Legendre factors. Results are summarized in Figure 4. All the time series strongly rejected the null hypothesis of unit root, indicating the existence of stationarity on the Legendre factors. On the other hand, the p-values reported in the BDS test indicate the existence of time dependence in the series of the translation factors of both markets, and in the series of the Global torsion factor.

We adjusted linear models to the time series of the Legendre factors. Figure 5 presents values for the R^2 of these models. Observe the low power of explanation of the linear models, with R^2 varying between 0% and 10%, for the one-dimensional models and between 9% and 38% for the VAR Model. This probably occurs because the biggest part of the dynamics is explained by independent terms. The repetition of the BDS test in the residuals of the linear adjusted models indicates the existence of some kind of non-linear dependence (in the mean) in the series of the translation factors of both markets and also in the series of the Global torsion factor.

In a qualitative exercise for identification of the main kinds of movements of the Brazilian sovereign term structures, we discarded information on the auto-covariance function of the Legendre factors process, and concentrated just in the covariance matrix of these factors. This is equivalent to discarding any type of time dependence among the factors. A motivation for this first approximation relies on the low power of explanation of the linear models presented in Figure 5, combined with the possibility of the existence of spurious non-linear dependence caused by illiquid bonds in the estimation process. Assumed the necessary simplifications, we applied our analysis of Section 3 to obtain approximated principal components and eigenvectors of the covariance matrix of the spread function.

We began defining an arbitrarily thin[10] uniform partition of the interval of maturities [0,40] years. Using this partition and Equation (9), we obtained the Legendre factors and estimated their sample covariance matrices for both markets: $\text{cov}(\hat{L}_{f\,G})$ and $\text{cov}(\hat{L}_{f\,B})$ [11]. We proceeded finding the spectral decomposition of these covariance matrices:

$$\text{cov}(\hat{L}_{f\,G}) = G_G \Gamma_G {G_G}^T \qquad (19)$$

[10] with 1000 points.
[11] We use subscripts *G* and *B* to denote respectively Global and Brady markets.

$$\text{cov}(\hat{L}_{f\,B}) = G_B \Gamma_B G_B{}^T \tag{20}$$

Figure 6 presents the estimated covariance matrices, and the orthogonal transformations G_G and G_B. Using these matrices and Equations (17) and (18), we are able to obtain the approximated eigenvectors and principal components of the spread functions for both markets:

$$fac_G(\tau) = G_G{}^T L_{f\,G}(\tau) \tag{21}$$

$$fac_B(\tau) = G_B{}^T L_{f\,B}(\tau) \tag{22}$$

$$\Psi_G = A_\xi G_G \tag{23}$$

$$\Psi_B = A_\xi G_B \tag{24}$$

Figures 7 and 8 present, respectively, the first three Legendre polynomials and the transformed Legendre polynomials given by Equations (23) and (24). Now, by the orthogonality of the principal components we are able to use their variances to identify the most important types of movements in the Brazilian sovereign term structures. These movements are to be elected among the transformed Legendre polynomials, the ones to which the principal components are attached. Figure 9 presents the covariance matrices of the principal components for both markets[12]. Observing Figures 8 and 9, we come to the following conclusions: The most important movement for the Global Term Structure is related to the translation factor, which explains 54.7% of the variability of the curve. Rotation movements capture 32.3% of the variability, while torsion movements are responsible for the remaining 13%. Regarding the Brady Bonds market, translations capture 61% of the movements, rotations 8%, and torsions 31%. This is a surprising result when compared to empirical studies realized with developed markets data. These studies usually identify the translation movements as responsible for something around 80% to 90% of the term structure movements

[12] We adopted the following order for the principal components: first factor, associated to translations, second factor associated to rotations, and third factor associated to torsions.

(Litterman and Scheinkman [1991], Barber and Copper [1996] and Singh [1997]), also indicating torsion movements as of low importance.

5. Conclusion

In this work, we presented a model to estimate non-observable term structures in risky markets. The model is an extension of the sectional model presented in Almeida et. al [1998], which captures the risk over a benchmark curve through a linear combination of Legendre polynomials. In its extension, the model is an affine factor model, which captures PCA as a particular case, and may be restricted to become arbitrage-free. Estimation process by use of Non-linear Kalman Filtering is briefly discussed, though replaced by sequential estimation to obtain time series for the non-observable term structures. In the numerical example, we studied the behavior of the Brazilian emerging market term structures. In particular, we found out that rotation and torsion movements appear to have more importance for emerging markets term structures than for developed countries term structures (at least when comparing our results to the ones documented in empirical works on developed markets term structures). In a forthcoming work, we intend to compare different techniques of estimation including the Kalman filter, sequential estimation and Efficient Method of Moments (Gallant and Tauchen [1996]).

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Figure 1. Sequentially Estimated Brazilian Term Structures of Brady Bonds and Global Bonds

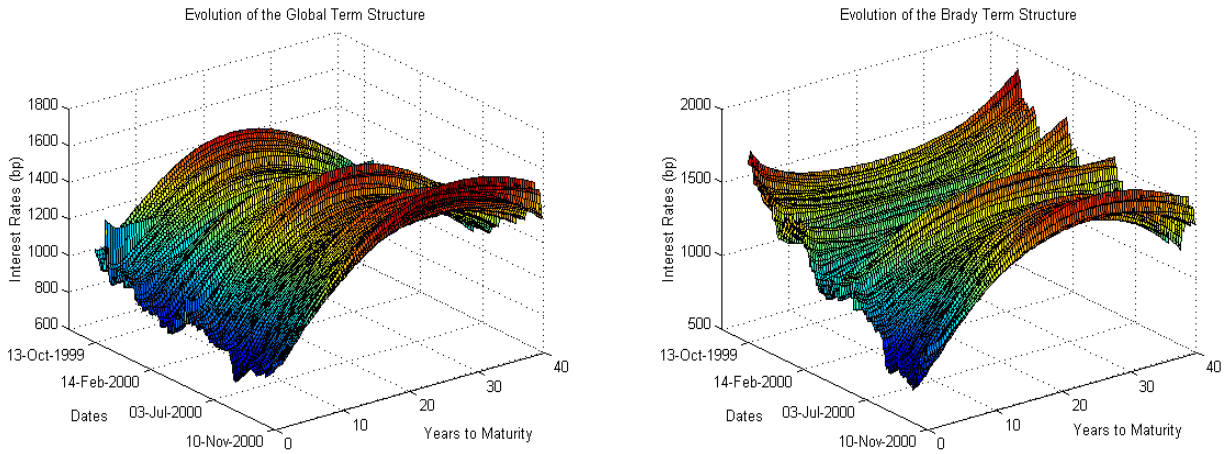

Figure 2. Time Series of the Legendre Coefficients

Figure 3. Time Series of the Legendre Factors

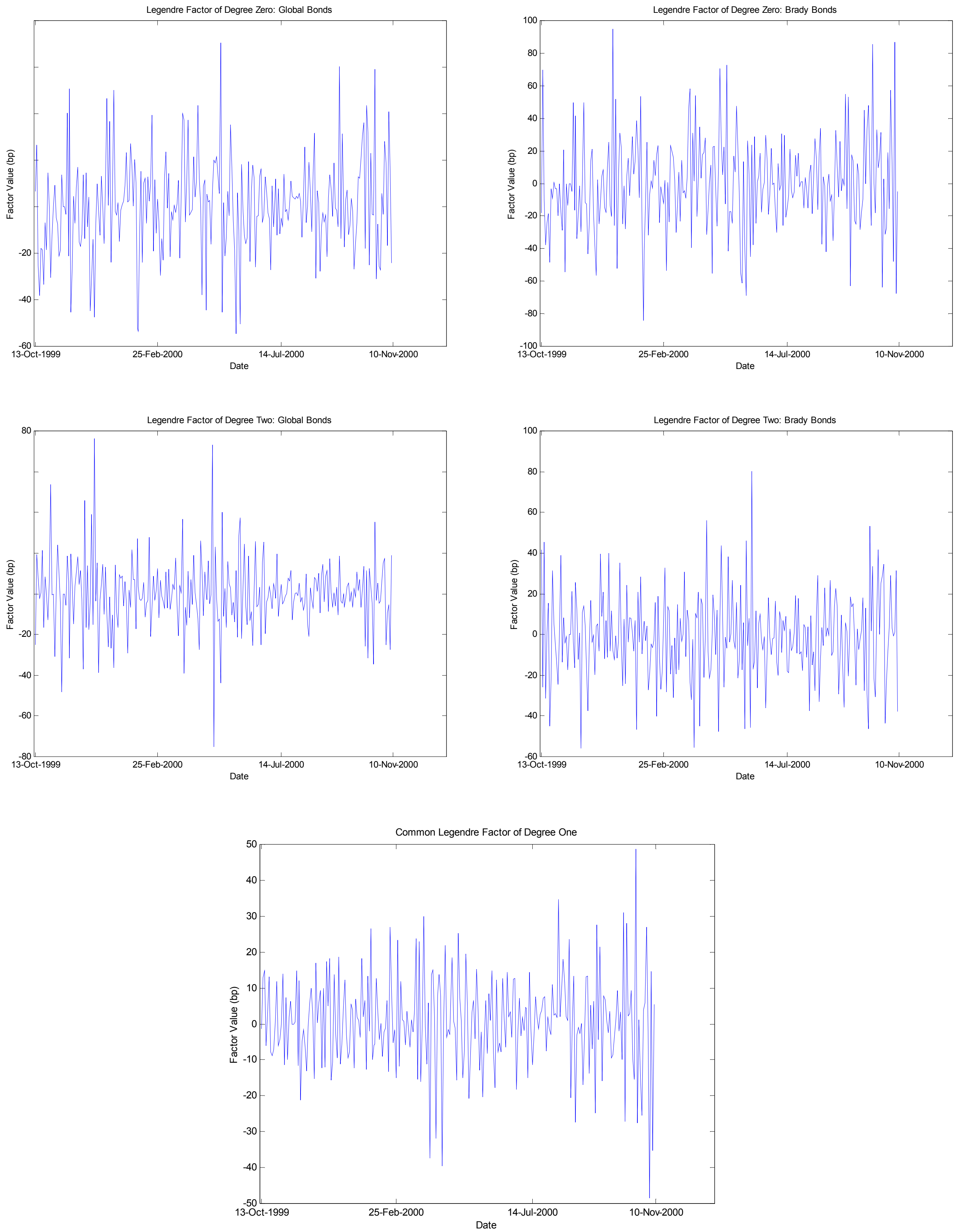

Figure 4. BDS and Unit Root Tests for the Legendre Factors

Legendre Factor	P-value BDS Test	P-value Dickey-Fuller
Translation – Global	0.02	< 0.01
Translation – Brady	0.05	< 0.01
Common Rotation	0.5	< 0.01
Torsion – Global	< 0.01	< 0.01
Torsion – Brady	0.45	< 0.01

Figure 5. Adjusted Linear Models to the Legendre Factors

Legendre Factor	ARMA Model	R^2 ARMA Model	R^2 VAR Model
Translation - Global	No correlation	0%	38%
Translation - Brady	No correlation	0%	13%
Common Rotation	MA(1)	4.8%	17%
Torsion - Global	ARMA(1,1)	7.8%	12%
Torsion - Brady	ARMA(1,1)	5.4%	9%

Figure 6. Covariance Matrices of Legendre Factors and Orthogonal Transformations

$$\operatorname{cov}(\hat{L}_{f\,G}) = 10^{-5} \cdot \begin{bmatrix} 0.4267 & 0.1101 & -0.0909 \\ 0.1101 & 0.1852 & 0.0326 \\ -0.0909 & 0.0326 & 0.2974 \end{bmatrix} \quad \operatorname{cov}(\hat{L}_{f\,B}) = 10^{-5} \cdot \begin{bmatrix} 0.8331 & 0.1861 & 0.0328 \\ 0.1861 & 0.1852 & 0.0785 \\ 0.0328 & 0.0785 & 0.4412 \end{bmatrix}$$

$$G_G = \begin{bmatrix} -0.4056 & 0.2033 & -0.8912 \\ 0.8408 & 0.4654 & -0.2765 \\ -0.3585 & 0.8614 & 0.3597 \end{bmatrix} \quad G_B = \begin{bmatrix} 0.9569 & 0.1661 & 0.2384 \\ 0.2660 & -0.1713 & -0.9486 \\ 0.1168 & -0.9711 & 0.2082 \end{bmatrix}$$

Figure 7. Three First Legendre Polynomials

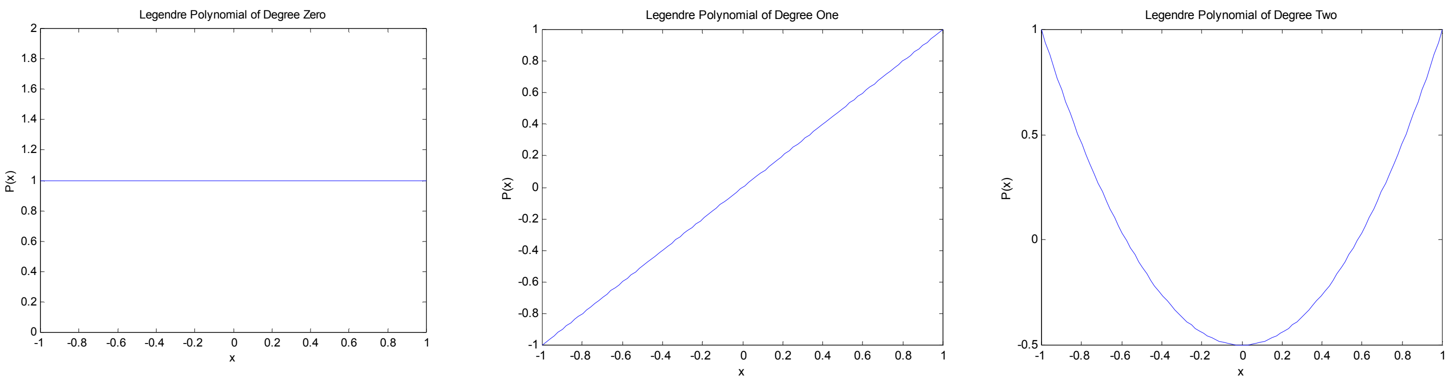

Figure 8. Transformed Legendre Polynomials

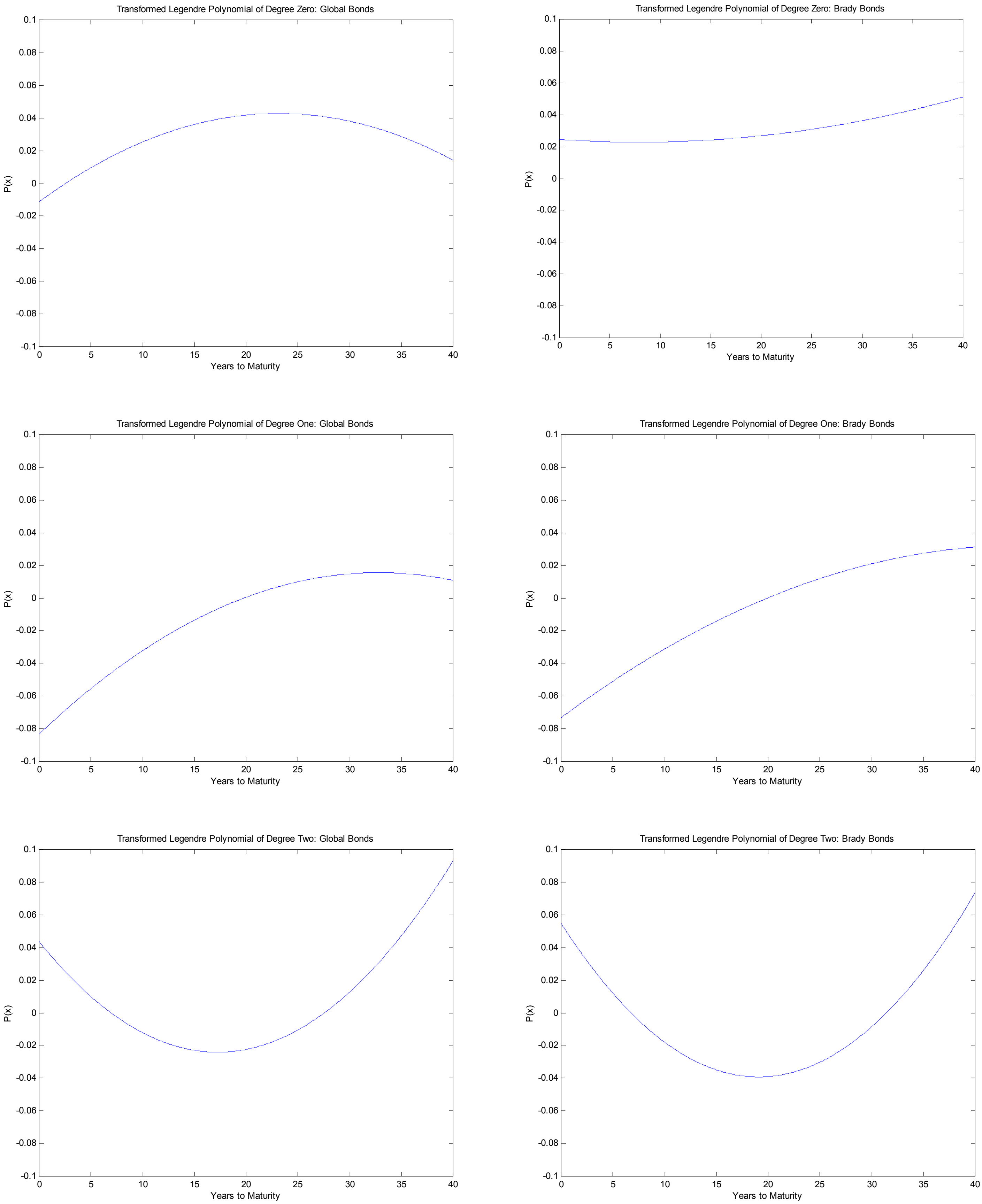

Figure 9. Covariance Matrices of the orthogonal factors fac_G and fac_B.

$$\mathrm{cov}(fac_G) = 10^{-5}\begin{bmatrix} 0.4975 & 0 & 0 \\ 0 & 0.2936 & 0 \\ 0 & 0 & 0.1182 \end{bmatrix} \quad \mathrm{cov}(fac_B) = 10^{-5}\begin{bmatrix} 0.8888 & 0 & 0 \\ 0 & 0.1212 & 0 \\ 0 & 0 & 0.4494 \end{bmatrix}$$