Online Appendix for: Economic Implications of Nonlinear Pricing Kernels

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Appendix A - Equivalence between CR $\gamma = 1$ and Hansen and Jagannathan (1991) with Nonnegativity Constraint

From expression (10) in Corollary 1 of the main paper, we obtain that our dual portfolio problem when $\gamma = 1$ will be:

$$\lambda^*_\gamma = \arg \sup_{\lambda \in \mathbb{R}^K} E \left[ \frac{a^2}{2} - \frac{1}{2} \left( a + \lambda' \left( R - \frac{1}{a} 1_K \right) \right)^2 I_{\Lambda_{CR}(R)}(\lambda) \right]$$  \hspace{1cm} (1)

where $\Lambda_{CR}(R) = \{ \lambda \in \mathbb{R}^K, \text{ s.t. } (a + \lambda' (R - \frac{1}{a} 1_K)) > 0 \}$, and $I(.)$ is an indicator function. First note that:

$$\left( a + \lambda' \left( R - \frac{1}{a} 1_K \right) \right)^2 I_{\Lambda_{CR}(R)}(\lambda) = \left( a + \lambda' \left( R - \frac{1}{a} 1_K \right) \right)^2 I_{\tilde{\Lambda}_{CR}(R)}(\lambda),$$ \hspace{1cm} (2)

where $\tilde{\Lambda}_{CR}(R) = \{ \lambda \in \mathbb{R}^K, \text{ s.t. } (a + \lambda' (R - \frac{1}{a} 1_K)) \geq 0 \}$. Second, note that we can get rid of the constant $\frac{a^2}{2}$, and of the number $-\frac{1}{2}$ that multiplies the term that depends on the Lagrange Multiplier vector $\lambda$, obtaining:

$$\lambda^*_\gamma = \arg \min_{\lambda \in \mathbb{R}^K} E \left[ \left( a + \lambda' \left( R - \frac{1}{a} 1_K \right) \right)^2 I_{\tilde{\Lambda}_{CR}(R)}(\lambda) \right]$$ \hspace{1cm} (3)

Now the trick is to note that, since $a > 0$ and, since we are searching for $\lambda \in \mathbb{R}^K$, we can divide $(a + \lambda' (R - \frac{1}{a} 1_K))$ by $a^2$, redefine $\tilde{\lambda} = \frac{\lambda}{a^2}$, and obtain the following equivalence for any $\lambda \in \mathbb{R}^K$:

$$I_{\tilde{\Lambda}_{CR}(R)}(\lambda) = I_{\tilde{\Lambda}_{CR}(R)}(\tilde{\lambda}),$$ \hspace{1cm} (4)

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where $\bar{\Lambda}_{CR}(R) = \{\lambda \in \mathbb{R}^K, \text{ s.t. } \left( \frac{1}{a} + \lambda' \left( R - \frac{1}{a}1_K \right) \right) \geq 0 \}$. Therefore (3) can be rewritten as:

$$\lambda^*_\gamma = \arg \min_{\lambda \in \mathbb{R}^K} a^4 E \left[ \left( \frac{1}{a} + \frac{\lambda'}{a^2} \left( R - \frac{1}{a}1_K \right) \right)^2 I_{\bar{\Lambda}_{CR}(R)}(\lambda) \right]$$

(5)

Which on its turn is equivalent to:

$$\lambda^*_\gamma = \arg \min_{\lambda \in \mathbb{R}^K} a^4 E \left[ \left( \frac{1}{a} + \lambda' \left( R - \frac{1}{a}1_K \right) \right)^2 I_{\bar{\Lambda}_{CR}(R)}(\bar{\lambda}) \right]$$

(6)

Getting rid of the $a^4$ term in front of the expectation, and noting that for any $\bar{\lambda} \in \mathbb{R}$, $\left( \frac{1}{a} + \bar{\lambda}' \left( R - \frac{1}{a}1_K \right) \right)$ is a return, we obtain an equivalent equation to problem P1 of Appendix A in Hansen and Jagannathan (1991).

**Appendix B - Taylor Expansion of the HARA Utility Function Implied by the Cressie Read Estimators**

For simplicity let us assume that there is only one risky asset with return $R$. According to the optimal portfolio interpretation section (subsection 2.2 of main paper), the utility function that is maximized to obtain the solution of the Cressie Read Bounds and their implied SDFs is given by:

$$u(v) = -\frac{1}{\gamma + 1} (1 - \gamma v)^{-\frac{\gamma + 1}{\gamma}}$$

(7)

where $v = \lambda^* \left( R - \frac{1}{a} \right)$, and $a$ represents the SDF mean. The solution of the HARA portfolio problem gives the optimal lambdas $\lambda_{opt}$ that will be used to define the Cressie Read bound and the corresponding implied SDF, both obtained at $v_0 = \lambda_{opt} \ast E[(R - \frac{1}{a})]$.

Now, we are interested in performing a Taylor expansion around the optimal $\lambda$-scaled expected excess return of the risky asset $v_0$ that will represent the aggregate risky in the economy. The goal is to analyze how the coefficient of risk aversion $\gamma$ will affect the weights given to skewness and kurtosis in the specific solutions of our HARA-utility problems. To that end, we use the corresponding second, third, and fourth derivatives of $u$ in a fourth order Taylor expansion, and take expected values of both sides:

$$E[u(v)] \approx u(v_0) + \frac{1}{2} u_2(v_0) \lambda_{opt}^2 \ast E(R - E(R))^2 + \frac{1}{6} u_3(v_0) \lambda_{opt}^3 \ast E(R - E(R))^3 + \frac{1}{24} u_4(v_0) \lambda_{opt}^4 \ast E(R - E(R))^4$$

(8)

Those derivatives are respectively given by:

$$u_2(v) = -(1 - \gamma v)^{-1 + \frac{1}{\gamma}}$$

(9)

$$u_3(v) = (1 - \gamma)(1 - \gamma v)^{-2 + \frac{1}{\gamma}}$$

(10)
Looking at the third derivative of $u$ we see that skewness could be weighed negatively for Cressie Read estimators with $\gamma > 1$. However, according to the Taylor expansion, the optimal lambda gives an extra degree of flexibility for the sign of the third moment. For instance, a negative lambda for estimators with $\gamma > 1$ will provide a positive weight to skewness. This flexibility guarantees that for the whole range of $\gamma$s the dual utility functions can potentially satisfy decreasing absolute risk aversion from Arditti (1967) ($\lambda^3_{opt} u_3(v0) > 0$). However, as stressed in section 2.4 of the main paper, decreasing absolute risk aversion can only be guaranteed in the region where $\gamma < 1$.

In Figure 1 of the main paper we provide pictures with the sensitivity of our estimators to skewness and kurtosis. They plot the third and fourth derivatives as functions of $\gamma$. Note that we chose optimal lambdas compatible with decreasing absolute risk aversion. The derivative functions are depicted for small positive, zero, and small negative $v_0$, which correspond to the lambda-scaled expected excess return of the risky asset. As in principle $v_0$ may achieve any arbitrary value being a solution to the HARA portfolio problem, it becomes clear the richness with which the CR estimators can weight skewness and kurtosis.

Note that skewness weights are always positive and they increase when $\gamma$ goes away from the quadratic case ($\gamma = 1$). For instance, EL ($\gamma = -1$) puts higher weights than ET ($\gamma = 0$), CUE (CR($\gamma = 1$)) gives zero weight, and CR($\gamma = 3$) gives weights comparable to EL ones. Regarding the fourth derivative, except for the region of $0.5 < \gamma < 1$, kurtosis is a non-positive and concave function of $\gamma$ indicating that all CR estimators outside that region satisfy the concept of decreasing absolute prudence proposed by Kimball (1993) ($\lambda^4_{opt} u_4 < 0$). Limiting cases including the quadratic utility (CUE, $\gamma = 1$) and the cubic utility (CR $\gamma = 0.5$) put zero weight to kurtosis. Note that Cressie Read estimators with positive $\gamma$’s give more (negative) weight to kurtosis than the corresponding estimators with negative $\gamma$’s. For instance, EL ($\gamma = -1$) weights kurtosis on the interval $[-12,-10]$, while CR($\gamma = 3$) weights it on the interval $[-20,-10]$, for the particular values of $v_0$ that we chose.

Appendix C - Expressions for the Components of the Discrepancy Measure of the Long-Run Risk models

We write below the expressions for the mean and variance components of the discrepancy formula in terms of the parameters of the long-run risk models:

$$E[M^s] = \exp \left\{ s \left[ \theta \log \delta - \frac{\theta}{\psi} \mu + (\theta - 1) \left\{ \kappa_0 + (\kappa_1 - 1)(A_0 + A_2 \sigma^2) + \mu \right\} \right] \right\} \cdot \exp \left\{ \frac{1}{2} s^2 \left[ \frac{\theta^2}{\psi^2} \sigma^2 + (\theta - 1)^2[(1 + B^2)\sigma^2 + (A_2 \kappa_1)^2 \sigma_w^2] - 2\frac{\theta}{\psi}(\theta - 1)\sigma^2 \right] \right\}$$

where: $B = \kappa_1 A_1 \psi e$. 

\[ u_4(v) = -(1 - \gamma)(1 - 2\gamma)(1 - \gamma v)^{-3 + \frac{1}{\gamma}} \] (11)
and

\[ A_0 = \frac{1}{1 - \kappa_1} \left[ \log \delta + \kappa_0 + (1 - \frac{1}{\psi}) \mu + \kappa_1 A_2 (1 - \gamma_1) \sigma^2 + \frac{\theta}{2} (\kappa_1 A_2 \sigma_w)^2 \right] \]

\[ A_1 = \frac{1 - \frac{1}{\psi}}{1 - \kappa_1 \rho} \]

\[ A_2 = -\frac{(\gamma - 1)(1 - \frac{1}{\psi})}{2(1 - \kappa_1 \nu_1)} \left[ 1 + \left( \frac{\kappa_1 \varphi_\epsilon}{1 - \kappa_1 \rho} \right)^2 \right] \]  

(12)

**Appendix D - Algebraic Details on the Demand Side Model**

Here, we present the calculations that are the basis to define the discrepancy of the demand model.

For simplicity let us divide our calculations in two parts.

\[ \mathcal{E} = \theta \log(\delta) - \frac{\theta}{\psi} \mu + (\theta - 1) (\kappa \epsilon_0 + \kappa \epsilon_1 \epsilon - \epsilon + \mu) \]  

(13)

\[ V_1 = \theta^2 \left( \frac{\sigma_\lambda}{1 - \rho^2} + \sigma_\eta \right) \]

\[ V_2 = \left( \frac{\theta}{\psi} \right)^2 \left( \frac{\alpha \epsilon \sigma_w}{1 - \nu^2} + \pi \epsilon \lambda + \sigma_\epsilon^2 \right) \]

\[ V_3 = (\theta - 1)^2 (V_{3,1} + V_{3,2} + V_{3,3} + V_{3,4} + V_{3,5} + V_{3,6}) \]  

(14)

\[ C_1 = 2\theta (\theta - 1) \left( \frac{A_1 \kappa_1 \epsilon_1 \epsilon_0 \sigma_w}{1 - \nu^2} - \frac{A_1 \sigma^2_w}{1 - \nu^2} - A_2 \sigma_\eta \right) \]

\[ C_2 = -\frac{\theta}{\psi} (\theta - 1) \left( \alpha_\epsilon^2 \left( \frac{\sigma_w^2}{1 - \nu^2} \right) + \pi \epsilon \lambda + \sigma_\epsilon^2 + \kappa \epsilon_1 \pi \epsilon_3 \sigma_\lambda + (\kappa \epsilon_1 - \nu) \left( \frac{A_3 \alpha \epsilon \sigma_w^2}{1 - \nu^2} \right) \right) \]

Where:
$$V_{3,1} = \kappa_{c1}^2 \left( A_{c1}^2 \frac{\sigma_\lambda^2}{1 - \rho^2} + A_{c2}^2 + A_{c3}^2 \frac{\sigma_\omega^2}{1 - \upsilon^2} \right)$$

$$V_{3,2} = A_{c1}^2 \frac{\sigma_\lambda^2}{1 - \rho^2} + A_{c2}^2 + A_{c3}^2 \frac{\sigma_\omega^2}{1 - \upsilon^2}$$

$$V_{3,3} = \alpha_{c}^2 \frac{\sigma_\lambda^2}{1 - \rho^2} + \pi c + \sigma_c^2$$

$$V_{3,4} = -2\kappa_{c1} \left( A_{c1}^2 \frac{\sigma_\lambda^2}{1 - \rho^2} + A_{c2}^2 \frac{\sigma_\omega^2}{1 - \upsilon^2} \right)$$

$$V_{3,5} = 2\kappa_{c1} \left( A_{c3} \alpha_{c} \frac{\sigma_\omega^2}{1 - \upsilon^2} + A_{c1} \sigma_\lambda \pi \right)$$

$$V_{3,6} = -2 \left( A_{c3} \alpha_{c} \upsilon \frac{\sigma_\omega^2}{1 - \upsilon^2} \right)$$

Thus

$$E[M^n] = \exp(sE)exp \left( \frac{1}{2} s^2 (V_1 + V_2 + V_3 + C_1 + C_2) \right)$$

(15)

Appendix E - Limiting HARA Utility Cases for $\gamma = -1$ and $\gamma = 0$

To obtain the logarithmic and exponential limiting cases, we adopt the translated utility:

$$\frac{a^\gamma + 1}{\gamma + 1} \left( a^\gamma - \gamma W \right)^{\frac{\gamma + 1}{\gamma}}, \tag{17}$$

exactly as it appears in Corollary 1, and make use of L’hopital’s rule. Now, we will show why one can use L’hopital’s rule. To that end, let:

$$f(\gamma) = (a^\gamma - \gamma W)^{\frac{\gamma + 1}{\gamma}}, \tag{18}$$

$$g(\gamma) = a^\gamma + 1 - f(\gamma), \tag{19}$$

$$h(\gamma) = \gamma + 1. \tag{20}$$

Note also that:

$$f'(\gamma) = -f(\gamma) \frac{1}{\gamma^2} ln(a^\gamma - \gamma W) + \frac{\gamma + 1}{\gamma} \left( \frac{ln(a) a^\gamma - W}{a^\gamma - \gamma W} \right), \tag{21}$$

$$g'(\gamma) = ln(a) a^\gamma + 1 - f'(\gamma), \tag{22}$$

$$h'(\gamma) = 1 \tag{23}$$

First, consider $\gamma = -1$, and note that $lim_{\gamma \to -1} g(\gamma) = 0$ and $lim_{\gamma \to -1} h(\gamma) = 0$, indicating that we can apply L’hopital’s rule obtaining:

$$lim_{\gamma \to -1} \frac{a^\gamma + 1}{\gamma + 1} = \frac{1}{\gamma + 1} (a^\gamma - \gamma W)^{\frac{\gamma + 1}{\gamma}} = lim_{\gamma \to -1} \frac{g(\gamma)}{h'(\gamma)} = ln(a) - f'(-1) = ln(1 + aW) \tag{24}$$
For the $\gamma = 0$, we note that the limit of (17) when $\gamma \to 0$ becomes $a - \lim_{\gamma \to 0} f(\gamma)$. Then, rewriting:

$$f(\gamma) = e^{\frac{\ln(a^\gamma - \gamma W)}{\gamma + 1}}$$

we also note that we can apply L’hopital’s rule in the exponent of the exponential function. Observing that $\ln (a^\gamma - \gamma W)' = \frac{a^\gamma \ln(a) - W}{(a^\gamma - \gamma W)}$ and $\left(\frac{\gamma}{\gamma + 1}\right)' = \frac{1}{(\gamma + 1)^2}$, we obtain:

$$\lim_{\gamma \to 0} f(\gamma) = e^{\ln(a) - W} = ae^{-W}$$

implying that the limiting utility will be $a - ae^{-W}$. 
References


### Table 1: Parameters Used in the Calibration of the Demand Model of Albuquerque, Eichenbaum and Rebelo (2012)

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### Table 2: Parameters Used in the Calibration of the Disappointment Aversion Model.

This table presents the preference parameters used in the calibration procedure of the Disappointment Aversion Model of Routledge and Zin (2010). The values for β, δ, and θ are from Bonomo, Garcia, Meddahi, and Tedongap (2011). We choose α = ρ = 1 to consider the simplest version of the model of Routledge and Zin (2010).

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