

Appendix of the paper: Are interest rate options important for the assessment of interest rate risk?[☆]

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Abstract

This Appendix is intended to complement the published version of the paper in the Journal of Banking and Finance. It is divided in two parts. Appendix A describes the different backtesting procedures adopted in the paper to test the dynamic term structure models. Appendix B presents in detail the methodology adopted to estimate the dynamic models.

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Appendix A - Backtesting Procedures

In this appendix we describe the backtesting procedures proposed by Kupiec (1995), Christoffersen (1998) and Seiler (2006), as well as the Portmanteau test developed by Ljung-Box (1978).

Suppose that we have a time series of VaRs with confidence levels of $1 - \alpha$ and a time series of profits and losses (P&L) on a portfolio over a fixed time interval. A violation occurs when P&L is lower than VaR. Each variable $I_t(\alpha)$ in the hit sequence is equal to zero if there is no violation at time t or equal to one if there is a violation at time t , viz:

$$I_t(\alpha) = \begin{cases} 1 & \text{if } P\&L_t \leq VaR_{t-HP,t} \\ 0 & \text{else.} \end{cases}$$

First we consider the single-period case (1-day ahead quantile forecasting). Christoffersen (1998) points out that the problem of determining the accuracy of a VaR model is equivalent to the problem of determining whether the hit sequence satisfies the following two properties:

1. *Unconditional Coverage Property*, which means that the average number of violations is statistically equal to the expected one. Formally $\mathbb{P}[I_{t+1}(\alpha) = 1 | \mathcal{F}_t] = \alpha$.
2. *Independence Property*, which means that violations are independently distributed. In other words, the history of VaR violations does not contain information about future violations. Formally $I_t(\alpha)$ and $I_{t+k}(\alpha)$ are independent for all t and $k \geq 1$.

The test developed by Kupiec (1995) examines only the unconditional coverage while Christoffersen (1998) considers both the unconditional coverage and independence. The number of violations observed in a sample of size L is

$$N_L(\alpha) = \sum_{t=1}^L I_t(\alpha).$$

If the model is accurate, N_L must present a binomial distribution. Using the Likelihood-Ratio test, the Kupiec's test statistic takes the form

$$2\ln \left(\left(\frac{1 - \alpha}{1 - \hat{\alpha}} \right)^{L - N_L(\alpha)} \left(\frac{\alpha}{\hat{\alpha}} \right)^{N_L(\alpha)} \right) \sim \chi^2(1), \quad \text{as } L \rightarrow \infty,$$

where $\hat{\alpha} = N_L(\alpha)/L$ and $\chi^2(\nu)$ stands for a chi-squared distribution with ν degrees of freedom.

The Kupiec test provides a necessary condition to classify a VaR model as adequate. But, not considering the arrival times of violations, it does not exclude models that do not satisfy the independence property. In other words, the Kupiec test can accept a model that creates successive violation clustering. In order to solve this problem, Christoffersen (1998) proposed a Markov test that takes into account the independence property. Within his test, if the

probability of a violation in the current period has no impact on the probability of violation in the next period, then it must hold that

$$\frac{L_{10}}{L_{00} + L_{10}} = \frac{L_{11}}{L_{01} + L_{11}},$$

where L_{ij} is the number of observations of I_t with value i followed by j , $i, j = 0, 1$. Again, the test statistic is given by the Likelihood-Ratio test:

$$2\ln \left(\frac{(1 - \pi_{01})^{L_{00}} \pi_{01}^{L_{01}} (1 - \pi_{11})^{L_{10}} \pi_{11}^{L_{11}}}{(1 - \pi)^{L_{00} + L_{10}} \pi^{L_{01} + L_{11}}} \right) \sim \chi^2(1), \quad \text{as } L \rightarrow \infty, \quad (1)$$

where $\pi_{ij} = \frac{L_{ij}}{\sum_j L_{ij}}$ and $\pi = \frac{L_{01} + L_{11}}{L}$. Note that this test does not depend on the true coverage value α , and thus, only tests the independence property. Indeed, Christoffersen (1998) also presents a test which simultaneously examines the unconditional coverage and the independence properties. We do not use this joint test because it does not disentangle which property has been violated.

Christoffersen's test considers only dependence from day to day, what implies that the independence property is not fully tested. In order to examine the existence of dependence with a time lag greater than one period, we can use the Ljung-Box test (see Ljung and Box (1978)). If $\hat{\gamma}_i$ is the empirical autocorrelation of order i , to test $\gamma_i = 0$ for the first K autocorrelations we can use the following statistic

$$L(L + 2) \sum_{i=1}^K \frac{\hat{\gamma}_i^2}{L - i} \sim \chi^2(K), \quad \text{as } L \rightarrow \infty. \quad (2)$$

In order to validate the accuracy of a 10-day VaR we have to adjust the independence property, because the quantile forecasts are subject to common shocks. This can be done by requiring that $I_t(\alpha)$ and $I_{t+k}(\alpha)$ are independent for all t and $k \geq 10$. This fact complicates the accuracy problem since the variables I_t are not presumed to be i.i.d. There are different ways to handle the dependence of lag lower than 10 (see Dowd (2007) and Seiler (2006)). The first alternative is ignoring dependence. Although it can appear to be a naive choice, ignoring dependence gives valuable information since it is equivalent to implementing a very conservative test¹. The problem is that we can reject "good" models. In others words, under this alternative probably the type I error is higher than the true type I error. Seiler (2006) and Dowd (2007) propose different tests that take into account dependence. Dowd (2007) uses a bootstrapping approach to yield an i.i.d. resample. On the other hand, Seiler's test considers that the model under analysis captures the true autocorrelations of lag lower than

¹The correlation between $I_t(\alpha)$ and $I_{t+k}(\alpha)$ for $k < 10$ are expected to be positive. Thus, the variance of N_L when ignoring dependence is lower than the variance when correlations are taken into account. Therefore the rejection region is larger if dependence is ignored.

10. Thus, based on an ARMA process, the dependence structure can be estimated from the hit sequence observations. If L is large enough, we have

$$N_L(\alpha) \sim \mathcal{N}(L\alpha, LA),$$

where \mathcal{N} represents a normal distribution and the correlation matrix A is estimated using only the hit sequences $I_t(\alpha)$ and $I_t(0.5)$ (see Seiler (2006) for details about this estimation procedure)².

The adjusted independence property can be validated using the test in (1) with the number of “00”, “01”, “10” and “11” in the hit sequence counted from 10 to 10 days, that is

$$L_{ij} = \# \{2 \leq t \leq L | I_t(\alpha) = i, I_{t+10}(\alpha) = j\} \text{ for } i, j = 0, 1.$$

Finally, to test autocorrelation with a time lag greater than one period we use the Ljung-Box test (2) with the summation extending from $i = 10$ to $i = K + 9$.

²We also experimented with the Bonferroni approach (see Dowd (2007)) to solve the problem of dependence, but the power of the resulting test was very low.

Appendix B - Maximum Likelihood Estimation

We adopt the Maximum Likelihood estimation procedure proposed by Chen and Scott (1993). We observe the following reference ID bonds yields through time: $rb_t(1)$, $rb_t(63)$, $rb_t(126)$, $rb_t(189)$, $rb_t(252)$, and $rb_t(378)$. Let rb represents the $H \times 6$ matrix containing these ID bonds yields for the whole time series of H points. Assume that model parameters are represented by vector ϕ and that the difference between times $t - 1$ and t is Δt . In both models, the relation between the yield of a reference ID bond with maturity τ and the state variables at time t is

$$R(t, \tau, \phi) = -\frac{A(\tau, \phi)}{\tau} - \frac{B(\tau, \phi)'}{\tau} X(t). \quad (3)$$

In accordance with the empirical term structure literature, we fix $N = 3$ and assume that the yields of the reference ID bond with maturities 1, 126 and 252 days are observed without error. As the state vector is three-dimensional, knowledge of functions $A(\tau, \phi)$ and $B(\tau, \phi)$ allows us to solve a linear system to obtain the values of the state vector at each time t :

$$\begin{aligned} rb_t(1) &= -\frac{A(1, \phi)}{1} - \frac{B(1, \phi)'}{1} X(t), \\ rb_t(126) &= -\frac{A(126, \phi)}{126} - \frac{B(126, \phi)'}{126} X(t) \quad \text{and} \\ rb_t(252) &= -\frac{A(252, \phi)}{252} - \frac{B(252, \phi)'}{252} X(t). \end{aligned} \quad (4)$$

For the reference ID bond with maturities 63, 189 and 378 days, we assume observation with Gaussian errors u_t uncorrelated along time:

$$\begin{aligned} rb_t(63) &= -\frac{A(63, \phi)}{63} - \frac{B(63, \phi)'}{63} X(t) + u_t(63), \\ rb_t(189) &= -\frac{A(189, \phi)}{189} - \frac{B(189, \phi)'}{189} X(t) + u_t(189), \\ rb_t(378) &= -\frac{A(378, \phi)}{378} - \frac{B(378, \phi)'}{378} X(t) + u_t(378) \quad \text{and} \end{aligned} \quad (5)$$

When options are taken into account in the estimation procedure, in addition to yields errors, we have to consider options errors defined by

$$u_t^{\text{call}} = c_t - cs_t, \quad (6)$$

where cs_t is the time t observed Black volatility for the call and c_t is the corresponding model implied Black volatility.

After extracting the corresponding state vector at the vector of parameters ϕ , we can write the log-likelihood function of the reference ID bond yields as

$$L(rb, \phi) = \sum_{t=2}^H \log p(X(t)|X(t-1); \phi) - (H-1) \log |jac| - \frac{H-1}{2} \log |\Omega| - \frac{1}{2} \sum_{t=2}^H u_t' \Omega^{-1} u_t, \quad (7)$$

where:

$$1. \quad jac = \begin{bmatrix} -\frac{B(1, \phi)'}{1} \\ -\frac{B(126, \phi)'}{126} \\ -\frac{B(252, \phi)'}{252} \end{bmatrix};$$

2. Ω represents the covariance matrix for u_t , estimated using the sample covariance matrix of the u_t 's implied by the extracted state vector along time;
3. $p(X(t)|X(t-1); \phi)$ is the transition probability density from $X(t-1)$ to $X(t)$ under measure \mathbb{P} . For the CIR model it will be a product of three noncentral chi-squares, and for the Gaussian model it will be a multivariate gaussian distribution.
4. $u_t' = [u_t(63) \ u_t(189) \ u_t(378)]$ when we do not use options in the estimation procedure and $u_t' = [u_t(63) \ u_t(189) \ u_t(378) \ u_t^{call}]$ when we use.

Our final objective is to estimate the vector of parameters ϕ which maximizes function $L(rb, \phi)$. In order to try to avoid possible local minima we use several different starting values and search for the optimal point by making use of the Nelder-Mead Simplex algorithm for non-linear functions optimization (implemented in the MATLAB *fminsearch* function) and the gradient-based optimization method (implemented in the MATLAB *fminunc* function).

References

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