Identifying Volatility Risk Premia from Fixed Income Asian Options *

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Abstract

Fixed income Asian options are frequently adopted by companies to hedge interest rate risk. Having a payoff structure depending on the cumulative short-term rate makes them particularly informative about interest rate volatility risk. Based on a joint dataset of bonds and Asian interest rate options, we study the inter-relations between bond and volatility risk premiums in a major emerging fixed income market. We propose and implement a dynamic term structure model that generates an incomplete market, compatible with a preliminary empirical analysis of the dataset. Approximation formulas for at-the-money Asian option prices avoid the use of computationally intensive Fourier transform methods, allowing for an efficient implementation of the model. The model generates bond risk premium strongly correlated (89%) with a widely accepted emerging market benchmark index (EMBI-Global), and a negative volatility risk premium, consistent with the use of Asian options as insurance in this market. Volatility premium explains a significant portion (33%) of bond premium, indicating that the Asian options market considerably affects the prices of risk of its neighbor bond market.

Keywords: Asian Options, Risk Premium, Dynamic Term Structure Models, Incomplete Markets.

JEL Codes: C13, G12, G13.

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1 Introduction

Interest rate Asian options[1] are frequently quoted by financial houses and largely adopted by banks and corporations to hedge financial costs (Chacko and Das (2002), Bakshi and Madan (2002)). They are attractive as cheaper alternatives to regular options such as caps, floors and collars, and their potential use as hedging instruments makes them particularly informative about risk premium. In fact, with a payoff structure directly depending on the integral of the short-term rate, they contain useful information on how investors perceive and price volatility risks[2]. But, how can we use such options to learn more about interest rate risks?

In this work, we try to answer this question, by analyzing the risk premium structure of bonds and Asian interest rate options through the lens of a dynamic term structure model. Risk premium is estimated from joint data on interest-rate Asian options and bond prices, and its behavior is analyzed through the implied stochastic discount factor that connects the two markets in the dynamic model[3].

Although the pricing of Asian options has tremendously developed with the recent Fourier inversion techniques proposed in Ju (1997), Bakshi and Madan (2000), and Chacko and Das (2002), the insertion of such options in the estimation process of a dynamic model remains unexplored. This is the first work that studies risk premium properties of Asian options, and we do so by providing efficient approximation formulas for at-the-money Asian option prices. Those analytical formulas prove useful in identifying volatility premium when the model is estimated. Their main advantage is to avoid

[1]Options that have payoffs depending on the average (or integral) of the short-term rate. Previous papers pricing fixed income Asian options include Geman and Yor (1993), Longstaff (1995), Leblanc and Scaillet (1998), Cheuk and Vorst (1999), Chacko and Das (2002), and Dassios and Nagaradjasarma (2003). For a theoretical assessment of Asian options based on a geometric Brownian motion process for the underlying asset, see for instance Carr and Schroder (2004), and Yor (1992), among others.

[2]In a recent paper, Carr et al. (2008) show under the Black and Scholes model, that arithmetic Asian call option prices increase with the level of volatility, and also with the length of duration of the sampled average (if the discounting effect is neglected). There they point out to the difficulties of obtaining this result even under a standard vanila model for the underlying asset.

[3]Other papers that estimated dynamic term structure models based on joint datasets of underlying assets and option prices include Longstaff et al. (2001), Bikbov and Chernov (2005), Almeida et al. (2006), Graveline (2006), Joslin (2007), and Han (2007).

an inversion of a Fourier or Laplace transform to obtain option prices, what is fundamental when extracting the state vector within the dynamic term structure model[4].

Note that correctly identifying volatility risk premium should be crucial to reconcile option implied volatilities with observed volatilities in spot markets (Pan (2002)). Thus, our efficient approach could be of direct use to risk managers in search of correctly marking to market interest rate risk factors appearing in integrated fixed income markets. Moreover, adopting equities data Chernov (2006) showed that a correct estimation of volatility risk premium reflects in better predictions of future volatility. Therefore, an efficient estimation of volatility risk premium appears to be also relevant to portfolio managers and policy makers since both are clearly interested in forecasting future volatility.

A preliminary empirical analysis of the joint dataset of bonds and at-the-money Asian options suggests that those options are not redundant and that volatility is an important source of incompleteness of the bond market. Based on this information, we propose an affine term structure model (Duffie and Kan (1996)) with unspanned stochastic volatility (USV; Collin Dufresne and Goldstein (2002b)) to analyze the risk premium structure of this joint dataset. In the proposed model, volatility of the short-rate is stochastic represented by a Cox et al. (1985) process (CIR process). The price of volatility risk is a time-varying process which implies that the term structure of volatility premiums is a joint function of the average cross section of bond yields and of the time series of at-the-money option prices. Volatility of the stochastic discount factor is represented by multiple sources of risk related to term structure movements and to the volatility of the short-term rate.

The incomplete market structure generated by unspanned stochastic volatility is strongly supported by innumerous studies, including Collin Dufresne and Goldstein (2002b), Heidari and Wu (2003), Li and Zhao (2006), Andersen and Benzoni (2005), Collin Dufresne et al. (2005), and Han (2007), among others. However, recently Joslin (2006) observed that a certain subset of USV affine models is not able to reproduce simultaneously the term structure of U.S. yield volatilities and implied volatilities of bond options. This inability is due to restrictions in the mean reversion rates of term structure

[4]For other studies using option price approximations in a fixed income context, see Collin Dufresne and Goldstein (2002a), Singleton and Umantsev (2003), and Schrager and Pelsser (2005), who all approximate coupon bond options and swaptions prices.

latent factors that appear in those models. Fortunately, those restrictions are attenuated under term structures with shorter maturities, which is the case of our dataset[5]. This allows our model to succeed in reproducing both bond volatilities and option implied volatilities observed in real data. Moreover, in order to further analyze the adequacy of the proposed model, we perform simulations of the economy implied by its dynamics. Our simulations generate bond and Asian option prices consistently reproducing the preliminary empirical results that motivated the adoption of an incomplete market model.

Empirical results indicate that bond risk premium[6] is positive during most of the sample period, and strongly correlated with an important benchmark for emerging markets debt premium, the EMBI-Global J.P. Morgan index. Model implied volatility perfectly captures the level of an EGARCH benchmark (see Figure 5), indicating that volatility risk premium is correctly estimated. Volatility risk premium is a negative and volatile time-varying process, consistent with results observed in equity and currency markets[7]. The positive covariation of volatility and the stochastic discount factor suggests that the Asian options work like insurance instruments, as previously suggested by Longstaff (1995), Bakshi and Madan (2002), and Chacko and Das (2002). In addition, volatility risk premium explains a significant portion of bond risk premium (negative correlation of 32.5%), a result related to Bollerslev and Zhou (2006) who find a variance risk premium explaining more than 15% of equity market portfolios excess returns.

Although many authors have studied volatility risk premium in the con-

[5]The dataset is composed by bonds and Asian interest rate options traded in a major emerging market: the Brazilian fixed income market. We explore the fact that in this market Asian options are regularly traded and officially offered by one of the biggest exchanges for futures and options in the world, the Brazilian Mercantile Futures Exchange (BM&F).

[6]The results presented are related to an arbitrarily fixed maturity of one year. Providing results to any other maturity would be immediate.

[7]Pan (2002), Bollerslev and Zhou (2002), Eraker (2004), Garcia et al. (2006), and Bollerslev et al. (2006) all obtain a negative volatility risk premium, when estimating variations of Heston's (1993) model with or without jumps in prices, and/or volatility. Chernov and Ghysels (2000) find a negative volatility risk premium (positive market price) for most of their sample period. Bakshi and Kapadia (2003) examine statistical properties of delta-hedged option portfolios and directly infer a negative volatility risk premium. On currency markets, Guo (1998) extracts a negative volatility premium, while Bates (1996) finds mixed evidence for the sign of volatility risk premium conditional on the kind of model adopted.

text of equity and currency markets, the same is not true for fixed income markets. Only two other papers also address this question in the context of interest rates. Fornari (2008) estimates the price of volatility risks from interest rate swaptions on Dollar, Euro, and Pound rates. Based on an asymmetric GARCH model, for all studied markets he finds a negative (and time-varying) volatility risk premium. Despite the similarity of results between his work and ours, the two methodologies adopted are quite distinct. While he extracts volatility premium directly from swaptions data only, we extract volatility premium from simultaneously bond and options data, under a continuous time dynamic term structure model that integrates the two markets. Closer in spirit to our paper, Joslin (2007) estimates different affine models based on joint data on U.S. bonds and swaptions, finding support for a negative volatility risk premium, also in line with our results. However, his empirical results support a class of weakly spanned volatility models as opposed to unspanned stochastic volatility models. In contrast, our results indicate that the USV model is able to fit well stylized facts of the joint bond / Asian option markets. Our distinguishing contribution relies in offering a parsimonious and computationally efficient arbitrage-free model to price the **volatility risk of Asian options**.

The rest of the paper is organized as follows. Section 2 describes the data adopted in the empirical analysis. Section 3 presents the dynamic model, and the pricing of zero-coupon bonds and Asian options. On Section 4, the model is estimated adopting a joint dataset on bonds and options, its ability to correctly price options is tested, and an analysis of model implied risk premia is provided. Section 5 concludes with some remarks and topics for future research. The Appendices contain technical information including proofs of lemmas, and calculations of bond conditional variances.

2 Data and Market Description

2.1 ID-Futures

The One-Day Inter Bank Deposit Future Contract (ID-Future) with maturity T is a future contract whose underlying asset is the accumulated daily ID rate[8] capitalized between the trading time t ($t \leq T$) and T. The contract

[8]The ID rate is the average one-day inter bank borrowing/lending rate, calculated by CETIP (Central of Custody and Financial Settlement of Securities) every workday. The

size corresponds to R$ 100,000.00 (one hundred thousand Brazilian Reals) discounted by the accumulated rate negotiated between the buyer and the seller of the contract.

This contract is very similar to a zero coupon bond, except that it pays margin adjustments every day. Each daily cash flow is the difference between the settlement price[9] on the current day and the settlement price on the day before corrected by the ID rate of the day before.

The Brazilian Mercantile and Futures Exchange (BM&F) is the entity that offers the ID-Future. The number of authorized contract-maturity months is fixed by BM&F (on average, there are about twenty authorized contract-maturity months for each day but only around ten are liquid). Contract-maturity months are the first four months subsequent to the month in which a trade has been made and, after that, the months that initiate each following quarter. Expiration date is the first business day of the contract-maturity month.

2.2 ID Index and its Option Market

The ID index (IDI) is defined as the accumulated ID rate. If we associate the continuously-compounded ID rate to the short term rate r_t then

$$IDI_t = IDI_0 \cdot \mathrm{e}^{\int_0^t r_u \mathrm{d}u}. \tag{1}$$

This index, computed on every workday by BM&F, has been fixed to the value of 100000 points in January 2, 1997, and has actually been resettled to its initial value a couple of times, most recently in January 2, 2003.

An IDI option with time of maturity T is an European option where the underlying asset is the IDI and whose payoff depends on IDI_T. When the strike is K, the payoff of an IDI option is $L_c(T) = (IDI_T - K)^+$ for a call and $L_p(T) = (K - IDI_T)^+$ for a put.

As can be noticed, IDI options have a peculiar characteristic which is not shared by usual fixed income international options: They are Asian options. Their payoff depends on the integral of the short-term rate through the path between the trading date t and the option maturity date T. As

ID rate is expressed in effective rate per annum, based on 252 business-days.

[9]The settlement price at time t of an ID-Future with maturity T is equal to R$ 100,000.00 discounted by its closing price quotation.

previously noticed, we will explore this payoff structure to extract volatility risk premium from the integrated bond/option fixed income market.

BM&F is also the entity that provides the IDI call options. Strike prices (expressed in index points) and the number of authorized contract-maturity months are established by BM&F. Contract-maturity months can happen to be any month, and the expiration date is the first business day of the maturity month. Usually, there are 30 authorized series within each day, from which about a third are liquid.

2.3 Data

Data consists on time series of yields of ID-Futures for all different liquid maturities, and values of IDI options for different strikes and maturities. The data covers the period from January 02, 2003 to December 30, 2005.

BM&F maintains a daily historical database with the price and number of trades of every ID-Future and IDI option that have been traded in any day. With the ID-Future database and a time series of ID interest rates, it is straightforward to estimate, by cubic interpolation, the interest rates for fixed maturities for all trading days. For each fixed time to maturity, a reference bond is a zero coupon bond with that corresponding time to maturity. We adopt the reference bond yields with fixed times to maturity of 1, 21, 42, 63, 126, 189, 252, 378 and 504 days. We also adopt at-the-money fixed-maturity IDI call options, whose prices were obtained via an interpolation based on black's implied volatilities[10].

After excluding weekends, holidays, and no-trade workdays, there is a total of 748 yields for each bond, and prices for at-the-money IDI call options[11].

[10]We fixed the option maturity to 95 days, the average maturity for available IDI call options. Our procedure is similar to that adopted to calculate VIX implied volatilities from S&P 500 index options for hypothetical at-the-money short-maturity (21 days) options. See Ewald et al. (2008) for a good discussion on how to interpolate implied volatilities from Asian options, and in particular, for an efficient way to compute those implied volatilities based on Monte Carlo methods.

[11]This sample size is compatible with that found in other recent academic studies containing derivatives data from emerging economies (see for instance, Pan and Singleton (2008)). In addition, as our study contains high frequency data, the number of observations (748) adopted to estimate the dynamic term structure model is large enough to avoid small-sample biases.

3 The Model

3.1 Evidence of Unspanned Stochastic Volatility in the Brazilian Market

Motivated by empirical results in Collin Dufresne and Goldstein (2002b), and Li and Zhao (2006), we investigate how well interest rates are able to explain call option prices, within the Brazilian fixed income market. As will be noted below, this relationship is useful when defining the probabilistic structure of the dynamic model.

We run regressions where the dependent variable is the price of the fixed-maturity at-the-money IDI call, while the independent variables are the yields of the reference bonds for the fixed maturities 21, 63, 126 and 252 days[12]. Let cs_t represent the time t price of the fixed-maturity at-the-money IDI call, and $rb_t(\tau)$ represent the time t yield of the reference bond with time to maturity τ, expressed in years. We basically run two types of regressions. The first, a standard multiple linear regression:

$$\begin{aligned} cs_t &= a_0 + a_1 rb_t(\tfrac{21}{252}) + a_2 rb_t(\tfrac{63}{252}) \\ &+ a_3 rb_t(\tfrac{126}{252}) + a_4 rb_t(1) + \epsilon_{t,1}. \end{aligned} \tag{2}$$

The second is a regression on a cubic polynomial form of the vector rb_t, with the four maturities chosen:

$$cs_t = a \cdot rb_t + b \cdot rb_t^2 + c \cdot rb_t^3 + \epsilon_{t,2}, \tag{3}$$

where a, b and c are four-dimensional vectors, and powers of rb_t's are calculated with operations simultaneously performed on each element of the vector.

R^2's, which represent the variability of call prices explained by bond yields, are respectively given by 0.18 and 0.35 for linear and polynomial regressions[13]. Similarly in spirit to the work of Collin Dufresne and Goldstein (2002b), results of these regressions suggest the existence of factors driving

[12] To avoid multicolinearity problems we do not use the maturities 42, 189 and 378 days. We experimented with different combinations of yields not reported here but that produced similar final results on the regression analysis.

[13] The polynomial regression can be seen as a multiple linear regression where the independent variables are the powers of the vector rb_t.

option dynamics with sources of uncertainty independent of the underlying market. In fact, since Vegas[14] of at-the-money options are extremely high, the calls used on the regressions above are mainly subject to volatility-risk[15]. It looks fair then to conclude that at least a substantial part of the variability of call prices unexplained by interest rates is due to the existence of extra factors driving volatility of interest rates[16]. In fact, after performing a simulation exercise based on parameters estimated using real data, we confirm that the proposed dynamic term structure model, whose incompleteness is due to the existence of USV, generates R^2's distributions with mean values very close to the values reported above (see Section 4.3).

After observing that volatility risk can not be completely hedged with bond portfolios in the Brazilian fixed income market, we adopt the framework presented in Casassus et al. (2005) to propose a four-factor dynamic term structure model where three factors are responsible for the cross section of bond prices and the fourth factor captures volatility of bond prices[17].

3.2 Model Specification and Bond Prices

The uncertainty in the economy is characterized by a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{P})$ where $(\mathcal{F}_t)_{t\geq 0}$ is the standard filtration generated by a four-dimensional Brownian motion $W^{\mathbb{P}} = \left(W_1^{\mathbb{P}}, \ldots, W_4^{\mathbb{P}}\right)$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$[18]. We assume the existence of a pricing measure $\mathbb{Q}$ under which discounted security prices are martingales with respect to $(\mathcal{F}_t)_{t\geq 0}$.

The model is within the class of affine models analyzed by Duffie and Kan (1996). It presents three stochastic factors, X_t, Y_t and Z_t that directly drive movements of the short term rate r_t, one stochastic factor v_t which represents the instantaneous volatility of factor X_t, and a conditionally deterministic

[14]The Vega of an option represents the first derivative of an option price with respect to the volatility state variable. See Hull (1990) for an explanation of the Greek variables.

[15]By the put-call parity, we know that an at-the-money straddle is equivalent to an at-the-money call, and the former is a portfolio mainly exposed to volatility-risk.

[16]Note that part of the unexplained variability of call prices might be due to other factors, like illiquidity (see Li and Zhao (2006) for a discussion).

[17]There is a fifth conditionally deterministic factor driving the short-rate long-term mean but not linked to any cross section instrument. Its role is to be a process auxiliary in generating USV constraints.

[18]See Duffie (2001) for details and formal definitions.

factor θ_t which represents the time varying long-term mean of factor X_t:

$$r_t = \phi_0 + X_t + Y_t + Z_t, \tag{4}$$

with:

$$\mathrm{d}X_t = \kappa(\theta_t - X_t)dt + \sqrt{v_t}\mathrm{d}W_X^{\mathbb{Q}}(t), \tag{5}$$

$$\mathrm{d}\begin{bmatrix} Y_t \\ Z_t \end{bmatrix} = -\overbrace{\begin{bmatrix} \eta_Y & 0 \\ 0 & \eta_Z \end{bmatrix}}^{\eta}\begin{bmatrix} Y_t \\ Z_t \end{bmatrix}\mathrm{d}t + \overbrace{\begin{bmatrix} \sigma_Y & 0 \\ \rho_{YZ} & \sigma_Z \end{bmatrix}}^{\sigma}\begin{bmatrix} \mathrm{d}W_Y^{\mathbb{Q}}(t) \\ \mathrm{d}W_Z^{\mathbb{Q}}(t) \end{bmatrix} \tag{6}$$

$$\mathrm{d}\theta_t = (\gamma - 2\kappa\theta_t + \frac{v_t}{\kappa})\mathrm{d}t \quad \text{and} \tag{7}$$

$$\mathrm{d}v_t = (\alpha - \beta v_t)\mathrm{d}t + \delta\sqrt{v_t}\mathrm{d}W_v^{\mathbb{Q}}(t); \tag{8}$$

where $W_X^{\mathbb{Q}}$, $W_Y^{\mathbb{Q}}$, $W_Z^{\mathbb{Q}}$ and $W_v^{\mathbb{Q}}$ are independent Brownian motions, and the volatility v_t follows a CIR (Feller) process[19]. By the independence assumption, if we condition on the path of volatility we have θ_t a deterministic function and the short rate would follow a three-factor extended Gaussian process with time-varying long term mean θ_t.

The assumption of independence among the sources of uncertainty represented by the Brownian motions at first glance seems restrictive. Nevertheless, some empirical studies (Ball and Torous (1999) and Heidari and Wu (2003)) find that innovations in interest rate levels are almost uncorrelated with innovations in the volatility of interest rates. We verify the validity of this assumption on the Brazilian fixed income market, by calculating the correlation between variations on the ID short-term rate and the volatility of the ID short-term rate variations estimated via a GARCH(1,1) scheme. The small absolute value of 1.5% for this correlation suggests that the independence assumption is acceptable in this context. In principle, the model could be extended to deal with correlation among the Brownian motions $W_Y^{\mathbb{Q}}$, $W_Z^{\mathbb{Q}}$ and $W_v^{\mathbb{Q}}$ or between $W_X^{\mathbb{Q}}$ and $W_v^{\mathbb{Q}}$ with no additional computational costs for

[19]A Feller process should satisfy the condition $2\alpha > \delta^2$ in order to guarantee its positivity. See Cox et al. (1985) for a univariate application in interest rates, and Heston (1993) for an example of a bi-dimensional log-affine process with a Feller process driving stochastic volatility.

bond pricing (see Casassus et al. (2005)). However, under correlated Brownian motions, the conditioning argument of Hull and White (1987) would not be valid anymore, and in principle we would lose the approximation option pricing formula[20].

Specification of a single factor driving the short-term rate volatility is another questionable feature. Although Andersen and Andreasen (2001) argue that only one factor can effectively capture the price dynamics of Bermudan swaptions, both Joslin (2006) and Andersen and Benzoni (2005) suggest the use of multiple factors driving volatility. In addition, Heidari and Wu (2003) show that three additional factors beyond level, slope and curvature are necessary to completely characterize swaption implied volatility surfaces. However, apart from their controversies, all these works agree on the existence of a main volatility factor describing about 80% of the dynamics of interest rates volatilities in U.S. fixed income markets. Favoring efficiency in the estimation process, since one volatility factor will capture most of the volatility dynamics and that it significantly simplifies the calculation of option prices, we adopt one volatility factor in our model. Our model is very similar to the one proposed by Collin Dufresne et al. (2005), which has been successfully applied to price U.S. swaps. Their four-factor USV term structure model simultaneously captures the cross section and short-rate volatility dynamics of the U.S swaps market.

It directly follows from Duffie and Kan (1996) and Casassus et al. (2005), that the time t price of a zero coupon bond maturing at time T is given by $(\tau = T - t)$:

$$P(t,T) = \mathrm{e}^{A(t,T)+B_X(\tau)X_t+B_Y(\tau)Y_t+B_Z(\tau)Z_t+B_\theta(\tau)\theta_t}, \tag{9}$$

where

$$B_X(\tau) = -\frac{1-\mathrm{e}^{-\kappa\tau}}{\kappa}, \tag{10}$$

$$B_Y(\tau) = -\frac{1-\mathrm{e}^{-\eta_Y\tau}}{\eta}, \tag{11}$$

$$B_Z(\tau) = -\frac{1-\mathrm{e}^{-\eta_Z\tau}}{\eta}, \tag{12}$$

[20]One possibility would be to price the options with Laplace/Fourier transform methods (Chacko and Das (2002)) therefore introducing higher computational costs to invert state variables from option prices. Alternatively, one could try to build on recent work by Alos and Ewald (2008) that provide an approximation formula for option prices under Heston's model, which avoids the use of Fourier invertion techniques.

$$B_\theta(\tau) = -\frac{(1-\mathrm{e}^{-\kappa\tau})^2}{2\kappa}, \tag{13}$$

$$A(t,T) = -\phi_0\tau + A_X(t,T) + A_{YZ}(t,T), \tag{14}$$

$$A_X(t,T) = -\frac{\gamma}{2\kappa}\left[\tau + \frac{1}{\kappa}\left(-\frac{3}{2} + 2\mathrm{e}^{-\kappa\tau} - \frac{\mathrm{e}^{-2\kappa\tau}}{2}\right)\right], \tag{15}$$

and

$$A_{YZ}(t,T) = \frac{\sigma_Y^2}{2\eta_Y^2}\left(B_Y(\tau)+\tau\right) - \frac{\sigma_Y^2}{4\eta_Y}B_Y(\tau)^2 + \tag{16}$$

$$\tag{17}$$

$$\frac{\rho_{YZ}^2+\sigma_Z^2}{2\eta_Z^2}\left(B_Z(\tau)+\tau\right) - \frac{\rho_{YZ}^2+\sigma_Z^2}{4\eta_Z}B_Z(\tau)^2. \tag{18}$$

Note that the price of the bond does not depend directly on the volatility variable v, creating an incomplete market where options are actually needed to hedge against the uncertainty of the volatility, not covered by the cross section of bond prices.

In order to relate the Brownian motions under the risk neutral measure to the Brownian motions under the objective measure, we have to define a parametric form for the risk premiums charged by investors. We work with a very general extended affine market price of risk (Cheridito et al. (2007)):

$$\mathrm{d}W_X^{\mathbb{Q}}(t) = \mathrm{d}W_X^{\mathbb{P}}(t) + \frac{1}{\sqrt{v_t}}\left(\lambda_0^X + \lambda_1^X X_t\right)\mathrm{d}t \tag{19}$$

$$\mathrm{d}W_{YZ}^{\mathbb{Q}}(t) = \mathrm{d}W_{YZ}^{\mathbb{P}}(t) + \sigma^{-1}\left(\lambda_0^{YZ} + \lambda_1^{YZ} YZ_t\right)\mathrm{d}t, \tag{20}$$

and

$$\mathrm{d}W_v^{\mathbb{Q}}(t) = \mathrm{d}W_v^{\mathbb{P}}(t) + \frac{1}{\delta\sqrt{v_t}}\left(\lambda_0^v + \lambda_1^v v_t\right)\mathrm{d}t, \tag{21}$$

where in Equation (20) we use the short notation:

$$\lambda_0^{YZ} = \begin{bmatrix} \lambda_0^Y \\ \lambda_0^Z \end{bmatrix} \quad \text{and} \quad \lambda_1^{YZ} = \begin{bmatrix} \lambda_1^Y & 0 \\ \lambda_{1,21}^{YZ} & \lambda_1^Z \end{bmatrix}.$$

Then, under the objective probability measure the dynamics of (X_t, Y_t, Z_t, v_t) is

$$\mathrm{d}X_t = \tilde{\kappa}\left(\tilde{\theta}_t - X_t\right)\mathrm{d}t + \sqrt{v_t}\mathrm{d}W_X^{\mathbb{P}}(t) \tag{22}$$

$$\mathrm{d}YZ_t = \tilde{\eta}\left(\tilde{\mu} - YZ_t\right)\mathrm{d}t + \sigma\mathrm{d}W_{YZ}^{\mathbb{P}}(t) \tag{23}$$

$$\mathrm{d}v_t = (\tilde{\alpha} - \tilde{\beta}v_t)\mathrm{d}t + \delta\sqrt{v_t}\mathrm{d}Z_t^{\mathbb{P}}, \tag{24}$$

where $\tilde{\eta} = \eta - \lambda_1^{YZ}$, $\tilde{\mu} = \tilde{\eta}^{-1}\lambda_0^{YZ}$, $\tilde{\kappa} = \kappa - \lambda_1^X$, $\tilde{\theta}_t = \frac{\kappa\theta_t + \lambda_0^X}{\tilde{\kappa}}$, $\tilde{\alpha} = \alpha + \lambda_0^v$, $\tilde{\beta} = \beta - \lambda_1^v$.

The risk neutral bond price dynamics is

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = r_t\mathrm{d}t + B_X(\tau)\sqrt{v_t}\mathrm{d}W_X^{\mathbb{Q}} + \left[B_Y(\tau)\ B_Z(\tau)\right]\sigma\mathrm{d}W_{YZ}^{\mathbb{Q}}, \tag{25}$$

Once more we can see that bond prices, on a short-term metric, are insensitive to volatility-risk and hence cannot be used to hedge it. Under the objective measure, the bond price dynamics is

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = (r_t + z^i(t,T))\mathrm{d}t + B_X(\tau)\sqrt{v_t}\mathrm{d}W_X^{\mathbb{P}} + \left[B_Y(\tau)\ B_Z(\tau)\right]\sigma\mathrm{d}W_{YZ}^{\mathbb{P}}, \tag{26}$$

where the instantaneous expected excess return is given by

$$z^i(t,T) = B_X(\tau)(\lambda_0^X + \lambda_1^X X_t) + \left[B_Y(\tau)\ B_Z(\tau)\right](\lambda_0^{YZ} + \lambda_1^{YZ}YZ_t). \tag{27}$$

3.3 Model Implied Conditional Volatility, and Volatility Risk Premium

From Equation (9) we know that the model implied yield at t for a time to maturity τ is given by

$$R(t,\tau) = -\frac{A(\tau)}{\tau} - \frac{B_X(\tau)}{\tau}X_t - \frac{B_Y(\tau)}{\tau}Y_t - \frac{B_Z(\tau)}{\tau}Z_t - \frac{B_\theta(\tau)}{\tau}\theta_t. \tag{28}$$

Note that it is an interesting empirical exercise to compare model-implied volatilities (or variances) to volatilities (variances) estimated via GARCH or EGARCH procedures, considered benchmark volatilities on previous empirical studies (see Dai and Singleton (2003) or Collin Dufresne et al. (2005)).

The model implied variance for a τ-maturity yield at $t+s$, conditioned on the information available until t is

$$var_t\left(R(t+s,\tau)\right) = \overline{B}(\tau)' var_t\left(E_{t+s}\right) \overline{B}(\tau), \tag{29}$$

where $E_t = (X_t, Y_t, Z_t, \theta_t, v_t)$ is the vector of state variables at time t,

$$\overline{B}(\tau) = -\begin{bmatrix} \frac{B_X(\tau)}{\tau} & \frac{B_Y(\tau)}{\tau} & \frac{B_Z(\tau)}{\tau} & \frac{B_\theta(\tau)}{\tau} & 0 \end{bmatrix}'.$$

and the variance of the state variables is taken under the objective probability measure P. Appendix B presents a simple algorithm to calculate the covariance matrix $var_t\left(E_{t+s}\right)$.

We know that within the class of affine models, the conditional yield variance is an affine function of the state variables, $var_t\left(R(t+s,\tau)\right) = b_0 + b_1 E_t$, where b_0 and b_1 are functions of $\overline{B}(\tau)$. In our model, vector b_1 has the first four elements equal to zero, which leads the dynamics of $var_t\left(R(t+s,\tau)\right)$ to be $\mathrm{d}var_t\left(R(t+s,\tau)\right) = b_{1,5}\mathrm{d}v_t$. As $\overline{B}(\tau)$ brings direct information about the average behavior of the cross section of bond yields, volatility, and consequently volatility risk premium, are mixed functions of the average behavior of the cross section of yields and the time series of v_t, which presents a direct correspondence to the time series of option prices in our model. In order to better illustrate this point, we consider the limiting case where s is close to zero, informally equivalent to an infinitesimal time dt. In this case, conditional volatility is given by

$$\begin{aligned} \frac{var_t(R(t+dt,\tau))}{dt} &= \left(\frac{B_Y(\tau)}{\tau}\right)^2(\sigma_y^2 + \rho_{YZ}) + \left(\frac{B_Z(\tau)}{\tau}\right)^2(\sigma_z^2 + \rho_{YZ}) + \\ + \left(\frac{B_X(\tau)}{\tau}\right)^2 v_t &= \tilde{b_0}(\overline{B}(\tau)) + \tilde{b_1}(\overline{B}(\tau))v_t \end{aligned} \tag{30}$$

Volatility risk premium, the difference between objective and risk-neutral conditional expectations of volatility, in this case would be given by:

$$\text{Vol premium}_t(\tau) = -\frac{\tilde{b_1}(\overline{B}(\tau))}{2\sqrt{var_t\left(R(t+dt,\tau)\right)}}(\lambda_0^v + \lambda_1^v v_t) \tag{31}$$

precisely a mixed function of $\overline{B}(\tau)$ and v_t as previously stated. Note from Equation (30) that as $\tilde{b_1}(\overline{B}(\tau)) > 0$, the signs of λ_0^v and λ_1^v determine the sign and behavior of volatility risk premium for all maturities. If both λ^v's are positive (negative), then volatility risk premium is negative (positive) for the whole sample. If λ^v's present different signs, volatility risk premium can switch signs across sample. In any case, volatility risk premium has the important characteristic of being a time-varying process, unless both λ^v's are null, in case it also becomes null.

3.4 Pricing IDI Options

It is important to emphasize that our approximation formula for IDI option prices strongly rely on the assumption of independence among the underlying Brownian motions[21]. With the independence assumption in mind, at time t, an IDI call with time of maturity T, and strike K can be priced by the same technique applied by Hull and White (1987): By the independence of the Brownian motions $W_X^{\mathbb{Q}}$, $W_Y^{\mathbb{Q}}$, $W_Z^{\mathbb{Q}}$ and $W_v{}^{\mathbb{Q}}$, conditioning on the volatility path does not affect the distribution of $W_X^{\mathbb{Q}}$, $W_Y^{\mathbb{Q}}$ and $W_Z^{\mathbb{Q}}$. Then we can use the law of iterated expectations to obtain the price as a double expectation. The inner expectation is going to present a Black and Scholes type of analytical formula, while the external expectation integrates the volatility distribution, essentially a non-central χ^2 distribution in our model.

In what follows, we present a series of lemmas helpful when obtaining the price of an IDI option as a function of the state variables in our model.

Let $\mathcal{F}_{t,T}^v$ be the σ-field that represents the information on the volatility process between times t and T, i.e $\mathcal{F}_{t,T}^v = \sigma\left\{v_u : u \in [t,T]\right\}$. Denote by $\mathcal{G}_{t,T}$ the σ-field generated by the union of the σ-fields $\mathcal{F}_{t,T}^v$ and $\mathcal{F}_t$, i.e. $\mathcal{G}_{t,T} = \sigma\left\{\mathcal{F}_t \cup \mathcal{F}_{t,T}^v\right\}$. The following lemma provides information about the distribution of the integral of the short-term rate, a fundamental variable when pricing interest rates Asian options.

Lemma 1 *Let $H(t,T) = \int_t^T r_u \mathrm{d}u$, where the r_t dynamics appears in Equation (4). Then conditional on $\mathcal{G}_{t,T}$, $H(t,T)$ is normally distributed with mean $M(t,T)$ and variance $V(t,T)$ given by:*

$$M(t,T) = \phi_0 \tau + M_X(t,T) + M_{YZ}(t,T)$$

and

$$V(t,T) = V_X(t,T) + V_{YZ}(t,T),$$

[21] Note however that recent work by Alos (2006) and Alos and Ewald (2008) extends the Hull and White (1987) formula by considering correlated Brownian motions. This extension opens the possibility for an attempt to obtaining an approximation formula for Asian options under correlated Brownian motions.

with

$$M_X(t,T) = -B_X(\tau)X_t + \int_t^T \left(1 - \mathrm{e}^{-\kappa(T-u)}\right)\theta_u \mathrm{d}u, \tag{32}$$

$$V_X(t,T) = \frac{1}{\kappa^2}\int_t^T \left(1 - \mathrm{e}^{-\kappa(T-u)}\right)^2 v_u \mathrm{d}u, \tag{33}$$

$$M_{YZ}(t,T) = -B_Y(\tau)Y_t - B_Z(\tau)Z_t \quad \textit{and} \tag{34}$$

$$V_{YZ}(t,T) = 2A_{YZ}(t,T), \tag{35}$$

where $A_{YZ}(t,T)$ *is given by Equation (16),* $v_u : u \in [t,T]$ *is the path of the volatility conditional on* $\mathcal{G}_{t,T}$ *and*

$$\theta_u = \mathrm{e}^{-2\kappa(u-t)}\left(\theta_t + \int_t^u \mathrm{e}^{-2\kappa(t-s)}\left(\gamma + \frac{v_s}{\kappa}\right)\mathrm{d}s\right), \quad t \le u \le T. \tag{36}$$

Lemma 2 *The time* t *price of a zero coupon bond maturing a time* T *can be written as*

$$P(t,T) = \mathrm{e}^{-\phi_0\tau - M(t,T) + \frac{V(t,T)}{2}},$$

that is,

$$\int_t^T \left(1 - \mathrm{e}^{-\kappa(T-u)}\right)\theta_u \mathrm{d}u = \frac{V_X(t,T)}{2} - \frac{A_X(\tau)}{2} + B_\theta(\tau)\theta_t. \tag{37}$$

Lemma 3 *The time* t *price of a call option on the IDI with time to maturity* T *and strike price* K *is*

$$c(t,T) = \mathrm{E}^{\mathbb{Q}}\left[f\left(IDI_t, K, t, T, V(t,T)\right)|\mathcal{F}_t\right], \tag{38}$$

where

$$f\left(IDI_t, K, t, T, V(t,T)\right) =$$

$$IDI_t\Phi(d) - KP(t,T)\Phi\left(d - \sqrt{V(t,T)}\right),$$

$$d = \frac{\log\frac{IDI_t}{K} - \log P(t,T) + V(t,T)/2}{\sqrt{V(t,T)}}, \tag{39}$$

and $\Phi(\cdot)$ *is the standard normal cumulative distribution function.*

If the call option is at-the-money (i.e., $IDI_t = KP(t,T)$) Equation (38) simplifies to

$$c(t,T) = \mathrm{E}^{\mathbb{Q}}\left[IDI_t \left(2\Phi\left(\frac{\sqrt{V(t,T)}}{2} \right) - 1 \right) \middle| \mathcal{F}_t \right]. \tag{40}$$

Using the fact that an at-the-money option is almost a linear function of Black's volatility (see Han (2007))[22], we obtain

$$c(t,T) = IDI_t \left[2\Phi\left(\frac{\sqrt{\mathrm{E}^{\mathbb{Q}}\left(V(t,T)|\mathcal{F}_t\right)}}{2} \right) - 1 \right]. \tag{41}$$

Lemma 4

$$\mathrm{E}^{\mathbb{Q}}\left(V(t,T)|\mathcal{F}_t\right) = V_{YZ}(t,T) + \frac{v_t}{\kappa^2} c_1(t,T) + \frac{\alpha}{\beta\kappa^2} c_2(t,T),$$

where:

- $c_1(t,T) = \frac{1-\mathrm{e}^{-\beta\tau}}{\beta} - 2\frac{\mathrm{e}^{-\beta\tau}-\mathrm{e}^{-\kappa\tau}}{\kappa-\beta} + \frac{\mathrm{e}^{-\beta\tau}-\mathrm{e}^{-2\kappa\tau}}{2\kappa-\beta}$ *and*
- $c_2(t,T) = \frac{1}{\kappa}\left(-\frac{3}{2} + 2\mathrm{e}^{-\kappa\tau} - \frac{\mathrm{e}^{-2\kappa\tau}}{2}\right) + \tau - c_1(t,T).$
- $V_{YZ}(t,T)$ *is given by Equation (35).*

Note that Lemma 3 completely characterizes the price of an IDI option as a function of the state variables $(X_t, Y_t, Z_t, \theta_t, v_t)$ while Lemma 4 combined with Equation (41) gives an approximation to the option price which depends only on the stochastic volatility variable v_t, as long as the option is at-the-money.

On empirical applications, the approximation proves to be very useful because stochastic volatility can be explicitly extracted from option prices, and the dynamic model can be optimized via a Quasi-Maximum Likelihood (QML) procedure, allowing the identification and posterior analysis of volatility risk premium. On the other hand, if we were interested in inversion of variable v_t from in- or out-of-the-money option prices, we could adopt a Taylor series expansion of the option price in Equation (38) combined to

[22]In fact, with the parameters estimated in Section 4, the error due to this approximation is smaller than 1% of the average at-the-money option price.

analytical results obtained for moments of the integrated volatility in affine models (see Garcia et al. (2006) and Bollerslev and Zhou (2002)) to again obtain approximations for the option prices as analytical functions of the state variables[23].

One last interesting point regards the validity of this approximation under an affine model not imposing the USV restriction. In this case, the approximation would still be valid whenever the stochastic volatility variable v is driven by one independent Brownian motion (${W_v}^{\mathbb{Q}}$). The bond price would be the product of the price provided on lemma 2 with the price of a bond under a CIR model where the short-term rate follows the dynamics of v. The at-the-money approximated option price would be unaltered.

3.5 Quasi-Maximum Likelihood Estimation

Let $rb_t(\tau)$ represent the time t yield of an ID reference bond with time to maturity τ years, and cs_t be the time t price of the at-the-money call option with time to maturity 95/252 years. We observe $rb_t(21/252)$, $rb_t(42/252)$, $rb_t(63/252)$, $rb_t(126/252)$, $rb_t(189/252)$, $rb_t(1)$ and $rb_t(378/252)$, for $t = 1, ..., N$. Denote by $IDIspot \in \mathbb{R}^N$ the vector of spot IDI. The parameter vector is given by $\phi = (\phi_0, \kappa, \gamma, \alpha, \beta, \delta, \eta_Y, \eta_Z, \sigma_Y, \sigma_Z, \rho_{YZ}, \lambda_0^X, \lambda_1^X, \lambda_0^Y, \lambda_1^Y, \lambda_0^Z, \lambda_1^Z, \lambda_1^{YZ}, \lambda_0^v, \lambda_1^v)$.

The state vector is inverted from observed data (yields and option prices) by following a procedure similar to Chen and Scott (1993). Four instruments are chosen to be priced without error: The at-the-money IDI call option with maturity of 95/252 years, and the yields of reference ID bonds with maturities 21/252, 63/252 and 1 year.

Reference ID bonds with maturities 42/252, 126/252, 189/252, and 378/252 years are assumed to be priced with Gaussian errors $u_t = [u_t(42/252), \dots, u_t(378/252)]'$ uncorrelated on the time scale.

The log-likelihood function is written by

[23] Note that if the issue is only pricing options as opposed to inverting state variables from option prices, the method proposed in Chacko and Das (2002) is appropriate for general affine models.

$$L(\phi, rb, cs, IDIspot) = \sum_{t=2}^{N} \log p\left((X_t, Y_t, Z_t, \theta_t, \upsilon_t)|(X_{t-1}, Y_{t-1}, Z_{t-1}, \theta_{t-1}, \upsilon_{t-1}); \phi\right) -$$

$$- \sum_{t=2}^{N} \log |Jac_t| - \frac{N-1}{2} \log |\Omega| - \frac{1}{2} \sum_{t=2}^{N} u_t' \Omega^{-1} u_t,$$

where:

1. Jac_t is the Jacobian matrix of the transformation defined by Equations (28) and (41), i.e.

$$Jac_t = \begin{bmatrix} -\frac{B_X(0.083,\phi)}{0.083} & -\frac{B_Y(0.083,\phi)}{0.083} & -\frac{B_Z(0.083,\phi)}{0.083} & 0 \\ -\frac{B_X(0.25,\phi)}{0.25} & -\frac{B_Y(0.25,\phi)}{0.25} & -\frac{B_Z(0.25,\phi)}{0.25} & 0 \\ -B_X(1,\phi) & -B_Y(1,\phi) & -B_Z(1,\phi) & 0 \\ 0 & 0 & 0 & g\Psi\left(\frac{\sqrt{\upsilon}}{2}, 0, 1\right) \end{bmatrix},$$

   with

   - $g = \frac{c_1(95/252,\phi) IDIspot_t}{2\kappa^2 \sqrt{\upsilon}}$,
   - $\upsilon = V_X(t,T) + \frac{\upsilon_t c_1(95/252,\phi)}{\kappa^2} + \frac{\alpha c_2(95/252,\phi)}{\beta\kappa^2}$,
   - c_1 and c_2 are given by Lemma 4, and
   - Ψ is the normal probability density function.

2. $p\left((X_t, Y_t, Z_t, \theta_t, \upsilon_t)|(X_{t-1}, Y_{t-1}, Z_{t-1}, \theta_{t-1}, \upsilon_{t-1}); \phi\right)$ is the transition probability from $(X_{t-1}, Y_{t-1}, Z_{t-1}, \theta_{t-1}, \upsilon_{t-1})$ to $(X_t, Y_t, Z_t, \theta_t, \upsilon_t)$, under the objective measure $\mathbb{P}$. Under the QML procedure it is approximated by a Gaussian distribution with known closed-form conditional mean and variance because the model is affine (see Appendices A and B in Duffee (2002)).

3. Ω is the covariance matrix for u_t, estimated using the sample covariance matrix of the u_t's implied by the extracted state vector for each point in time.

Our final objective is to estimate the vector of parameters ϕ maximizing function $L(\phi, rb, cs, IDIspot)$. To avoid possible local minima we use several different starting values and search for the optimal point by making use of Nelder-Mead Simplex / Gradient-based optimization methods.

4 Empirical Results

4.1 Parameters and Cross Section Pricing

The model was estimated using Quasi Maximum Likelihood, inverting the state vector from bond yields with maturities 21, 63, and 252 days, and one at-the-money IDI option, with time to maturity of 95 days. Table 1 presents parameter values and asymptotic standard deviations of parameters' estimators obtained by the BHHH method (Davidson and MacKinnon (1993))[24].

The small value of κ indicates that X, the only factor with stochastic volatility, is the term structure level factor[25]. The large value of η_Y shows that Y, the most volatile Gaussian factor, is a fast mean reverting factor. Figure 1 presents the time series of v, where we can observe different spikes, a pattern compatible with a fast mean reverting variable (high β). In addition, note the similar shape of v and black implied volatilities of the option used in the estimation process (compare Figures 1 and 2)[26].

Figure 3 presents the evolution of the Brazilian term structure of interest rates from January, 2003 to December, 2005. It presents a descending trajectory coming from interest rates higher than 25% in 2003 to lower values around 15% later in the sample. Figure 4 displays the observed and model implied term structures averaged across the sample. Note that the model approximately captures the average term structure, with a worse performance for the longest-maturity (1.5 years), whose average error is of 0.25%. Mean absolute errors for maturities (42, 126, 189, 378 days) priced with error are respectively 0.04%, 0.11%, 0.10%, and 0.33%. Standard deviations directly obtained from the time series of the errors are 0.06%, 0.12%, 0.11% and

[24]The third column in Table 1 displays the ratio *estimated parameter value* over *asymptotic standard deviation*. Values of this ratio above 1.96 appear in bold and indicate significant parameters, at a 95% confidence level. Parameters not shown in the table are zero valued.

[25]See Litterman and Scheinkman (1991).

[26]This is due to the approximation of an at-the-money option price as a linear function of Black's implied volatility.

0.35%. Those errors are comparable to results previously obtained in papers presenting dynamic term structure models estimated based on only bonds data[27]. Despite presenting a USV variable (v) not present in bond pricing formulas, the model is successful when pricing bonds because three factors (X, Y, Z) capturing the fundamental movements of the term structure (Litterman and Scheinkman (1991)) still appear within the cross section of bond prices[28].

4.2 Bond Risk Premium and Volatility Risk Premium

Figure 5 presents the one-year bond yield model implied historical volatility under the objective measure (dotted line), model implied risk neutral volatility (dashed line) and an EGARCH(1,1) volatility (solid line) estimated on daily variations of the same yield[29]. EGARCH volatility and model implied historical volatility appear on the same level, with a positive correlation coefficient of 45%[30]. Pan (2002) argues that volatility risk premium is crucial to relate option implied volatilities to observed volatilities in the spot market. Taking this observation into account, the fact that model implied volatility perfectly captures the level of historical volatilities would be a good indication that the volatility risk premium is correctly estimated and is important here to reconcile bonds and options dynamics.

Observing Figure 5 we note that the risk neutral volatility is much higher than the historical volatility, compatible with a **negative volatility risk premium**. Indeed, λ_0^v and λ_1^v are positive and statistically significant, what according to Section 3.3, indicates that volatility risk is negatively priced, consistent with the idea that volatility changes would be negatively correlated with aggregate consumption growth, in an equivalent economy generated with consumption data (see Guo (1998)). Figure 7 presents 1-year bond and volatility risk premiums. For the bond, risk premium can be interpreted as minus the covariance between its return and the Stochastic Discount Factor

[27]See Duffie and Singleton (1997) for an implementation with U.S. data, and Almeida (2004) for an implementation with Brazilian data.

[28]θ is extracted in a way to match USV constraints and is not inverted from the cross section of bonds, nor from option prices.

[29]We arbitrarily fixed the maturity at one year but the analysis could have been pursued for any other maturity.

[30]This result is comparable to that in Jacob and Karoui (2006) for short-term maturities, who obtained positive correlations of the order of 60% between EGARCH and model implied volatilities, for three-factor affine non-USV models.

(SDF) generated in our economy. For the volatility, risk premium has a similar interpretation but the covariance that should be considered is the one between volatility and the SDF. We directly observe that the covariance between volatility and the SDF is positive in the whole sample, while the covariance between bond returns and the SDF alternates between negative (mostly in the beginning and center of the sample) and positive periods. In periods where bond risk premium is negative, volatility premiums have values closer to zero. One possible interpretation for these results is that investors might be using options as insurance instruments, useful to protect against periods of low aggregate consumption (or consumption growth). As previously stated in Section 1, similar negative volatility risk premium was observed in many studies applied to equities and currency markets, where joint datasets of underlying/option data were adopted. One last interesting variable to observe is the correlation between volatility risk premium and the EGARCH historical volatility (-36.63%). This significant correlation is aligned, from a qualitative viewpoint, to Chernov's (2006) suggestions that correctly capturing volatility risk premium allows for an efficient use of option implied volatilities in predicting future volatility.

Figure 6 presents the 1-year model implied bond risk premium and the Global Emerging Markets Bond Index (EMBI-G) a J.P. Morgan index which included 27 emerging market countries in 1999 (see Cavanagh and Long (1999)), and is a established benchmark for emerging markets debt. Note that although this index was not adopted in the estimation process, and that it is related to global debt rather than to local debt, the Brazilian local term structure/options data used to estimate the dynamic model were enough to produce a risk premium strongly correlated to the EMBI-G, with a positive correlation coefficient of 89%. In addition, the correlation between the 1-year bond risk premium and the 1-year bond volatility risk premium is of -32.5%, indicating that volatility premium is an important portion of bond premium. This result is in line with the findings of Bollerslev and Zhou (2006) for the U.S. equity market, who extract for the S&P 500 index, variance premium from the difference between implied and realized variance, and show that it explains more than 15% of excess returns on the market portfolio over the 1990-2005 period.

4.3 Does the Model Reproduce the Regression Results that Motivated USV?

In this section, we want to identify if simulated data from the estimated model would generate results compatible with those obtained when we ran the regressions of Section 3.1, of at-the-money call option prices on a truncated power series of the underlying term structure of bonds.

The simulation exercise consisted in generating 1000 paths for the economy described by our model (with parameters appearing in Table 1) and running the regressions of Section 3.1 to obtain a distribution for the R^2's. Figure 5 presents histograms of the R^2's obtained for the linear and polynomial regressions. The R^2 mean value for the linear regression is 0.28, while for the polynomial regression it is 0.41. 95% confidence intervals for those R^2's coming from the simulation paths are $[0.03, 0.61]$ and $[0.17, 0.69]$. The R^2 values obtained with real data in Section 3.1, were respectively 0.18 and 0.35, indicating that the dynamics generated by our model produces bond yields and Asian option call prices compatible with the original data, and with the regressions that motivated the adoption of an USV model.

5 Conclusion

Asian options are known to be useful cheaper alternatives (to vanilla derivatives) to hedge interest rate risk. Exploring this fact, and interested in the risk premium structure of interest rate volatility, we study how informative a joint dataset of bonds and fixed income Asian options is with respect to the way investors perceive and price risk. We provide closed-form formulas for at-the-money Asian option prices, which allow for a **pioneering** implementation of a dynamic term structure model using **joint bond and Asian options data**. Our formulas allow us to avoid the more computationally intensive Fourier/Laplace transform inversion methods (see Duffie at all (2000), Bakshi and Madan (2000), or Chacko and Das (2002)), bringing efficiency to the optimization problem. Our model generates an incomplete market were bonds solely can not hedge volatility risk (Unspanned Stochastic Volatility, Collin Dufresne and Goldstein (2002b)). Volatility, and volatility risk premium are flexible time-varying processes identified through a combination of the average cross section of bonds and the time series behavior of option prices.

Based on a unique dataset of at-the-money Brazilian interest rate Asian options, in our model, bond implied risk premiums strongly correlate (89%) with an internationally accepted measure of Emerging markets risk premium (EMBI-G). Model implied volatility perfectly captures the level of an EGARCH historical volatility benchmark, and positively correlates (45%) with this measure. A negative value for the bond volatility risk premium is obtained implying that risk-averse investors are accepting to receive low (possibly negative) returns to hold the options (see Figure 5), and that options indeed work as insurance instruments to hedge bond risk. This result is in line with previous work on equity and currency markets (Chernov and Ghysels (2000); Guo (1998)), and also with more recent analysis of volatility risk premium in swaptions markets (Fornari (2008); Joslin (2007)). In addition, volatility risk premium explains 32.5% of bond premium, indicating that fixed income options should indeed be adopted (jointly with bonds) to identify how investors price risk in bond markets. A simulation exercise further confirms that the model implied dynamics correctly captures the portion of information carried in option prices that is spanned by bond prices, observed in real data.

It should be noticed that the whole estimation process was performed assuming that there were no additional factors (like illiquidity factors) segmenting the underlying and option markets, and also noticed that only at-the-money options were used to extract the latent factors time series. A possible alternative more robust model could consider incorporating jumps such as in Chacko and Das (2002) or Jiang and Yan (2006), with a richer set of options in the estimation, or incorporating illiquidity shocks in the spirit of Liu and Yong (2005).

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Appendix A - Proofs of Lemmas

Proof of Lemma 1
By definition of r_t we have

$$H(t,T) = \phi_0 \tau + \int_t^T X_u \mathrm{d}u + \int_t^T (Y_u + Z_u)\mathrm{d}u.$$

From Brigo and Mercurio (2001) we know that $\int_t^T (Y_u + Z_u)\mathrm{d}u$ conditioned on $\mathcal{F}_t$ is normal with mean and variance given by Equations 34 and 35 respectively[31].

Conditioning on the volatility path or equivalently, making v_t a deterministic function of time, it is not hard to verify by Ito's rule that for each $t < T$ the unique (strong) solution of (5) is

$$X_T = \Xi(T) \cdot \left[X_t + \kappa \int_t^T \Xi^{-1}(s)\theta_s \mathrm{d}s + \int_t^T \Xi^{-1}(s)\sqrt{v_s}\mathrm{d}s \right],$$

where $\Xi(x) = \mathrm{e}^{-\kappa(x-t)}$. Stochastic integration by parts implies that

$$\int_t^T X_u \mathrm{d}u = \int_t^T (T-u)\, \mathrm{d}X_u + (T-t)\, X_t. \tag{42}$$

By definition of process X_t, the integral in the right-hand side can be written as[32]

$$\int_t^T (T-u)\, \mathrm{d}X_u = \kappa \int_t^T (T-u)\,(\theta_u - X_u)\, \mathrm{d}u + \int_t^T (T-u)\, \sqrt{v_u}\mathrm{d}W_X(u).$$

But

$$\int_t^T (T-u)\, X_u \mathrm{d}u =$$

$$= \kappa \int_t^T (T-u)\, \Xi(u) \int_t^u \Xi^{-1}(s)\theta_s \mathrm{d}s\mathrm{d}u + X_t \int_t^T (T-u)\, \Xi(u)\mathrm{d}u +$$

$$+ \int_t^T (T-u)\, \Xi(u) \int_t^u \Xi^{-1}(s)\sqrt{v_s}\mathrm{d}W_X(s)\mathrm{d}u.$$

[31] We refer to Karatzas and Shreve (1991) for more details about solutions of stochastic differential equations.

[32] In this appendix we drop the superscript $\mathbb{Q}$ and denote the Brownian motion $W^{\mathbb{Q}}$ simply by W.

Calculating separately the last two integrals, we have

$$\int_t^T (T-u)\,\Xi(u)\mathrm{d}u = \frac{T-t}{\kappa} + \frac{\mathrm{e}^{-\kappa(T-t)}-1}{\kappa^2}$$

and, again by integration by parts,

$$\int_t^T (T-u)\,\Xi(u)\int_t^u \Xi^{-1}(s)\sqrt{v_s}\mathrm{d}W_X(s)\mathrm{d}u =$$

$$= \int_t^T \left(\int_t^u \Xi^{-1}(s)\sqrt{v_s}\mathrm{d}W_X(s)\right)\mathrm{d}_u\left(\int_t^u (T-s)\,\Xi(s)\mathrm{d}s\right) =$$

$$= \left(\int_t^T \Xi^{-1}(s)\sqrt{v_s}\mathrm{d}W_X(s)\right)\left(\int_t^T (T-s)\,\Xi(s)\mathrm{d}s\right) -$$

$$- \int_t^T \left(\int_t^u (T-s)\,\Xi(s)\mathrm{d}s\right)\Xi^{-1}(u)\sqrt{v_u}\mathrm{d}W_X(u) =$$

$$= \int_t^T \left(\int_u^T (T-s)\,\Xi(s)\mathrm{d}s\right)\Xi^{-1}(u)\sqrt{v_u}\mathrm{d}W_X(u)$$

Substituting the previous terms in Equation 42 and after some algebraic manipulation, we obtain

$$\int_t^T X_u\mathrm{d}u = \kappa\int_t^T (T-u)\theta_u\mathrm{d}u - \kappa^2 I(t,T) + \frac{1-\mathrm{e}^{-\kappa(T-t)}}{\kappa}X_t +$$

$$+ \int_t^T \left(T-u-\kappa\int_t^T (T-s)\mathrm{e}^{-\kappa(s-u)}\mathrm{d}s\right)\sqrt{v_u}\mathrm{d}W_X(u),$$

where $I(t,T) = \int_t^T (T-u)\int_t^u \mathrm{e}^{-\kappa(u-s)}\theta_s\mathrm{d}s\mathrm{d}u$. Then

$$M_X(t,T) = \kappa\int_t^T (T-u)\theta_u\mathrm{d}u - \kappa^2 I(t,T) + \frac{1-\mathrm{e}^{-\kappa(T-t)}}{\kappa}X_t, \tag{43}$$

Using Ito's isometry we have

$$V_X(t,T) = \frac{1}{\kappa^2}\int_t^T \left(1-\mathrm{e}^{-\kappa(T-u)}\right)^2 v_u\mathrm{d}u.$$

Now, changing the order of integration in $I(t,T)$ we obtain

$$I(t,T) = \int_t^T \frac{T-u}{\kappa}\theta_u \mathrm{d}u + \int_t^T \frac{\mathrm{e}^{-\kappa(T-u)}-1}{\kappa^2}\theta_u \mathrm{d}u.$$

Substituting this expression in Equation 43 we conclude the proof. □

Proof of Lemma 2
By Equation 36 we have

$$\int_t^T \left(1-\mathrm{e}^{-\kappa(T-u)}\right)\theta_u \mathrm{d}u =$$

$$\int_t^T \mathrm{e}^{-2\kappa(u-t)}\left(\theta_t + \int_t^u \mathrm{e}^{-2\kappa(t-s)}\left(\gamma+\frac{v_s}{\kappa}\right)\mathrm{d}s\right)\left(1-\mathrm{e}^{-\kappa(T-u)}\right)\mathrm{d}u =$$

$$\theta_t \mathrm{e}^{2\kappa t}\int_t^T \mathrm{e}^{-2\kappa u}\left(1-\mathrm{e}^{-\kappa(T-u)}\right)\mathrm{d}u +$$

$$\int_t^T \int_s^T \mathrm{e}^{-2\kappa u}\left(1-\mathrm{e}^{-\kappa(T-u)}\right)\mathrm{e}^{2\kappa s}\left(\gamma+\frac{v_s}{\kappa}\right)\mathrm{d}u\mathrm{d}s =$$

$$\frac{\theta_t}{2\kappa}\left(1-\mathrm{e}^{-\kappa(T-t)}\right)^2 + \frac{1}{2\kappa}\int_t^T \left(\gamma+\frac{v_u}{\kappa}\right)\left(1-\mathrm{e}^{-\kappa(T-u)}\right)^2 \mathrm{d}u =$$

$$B_\theta(\tau)\theta_t - A_X(t,T) + \frac{V_X(t,T)}{2},$$

which concludes the proof. □

Proof of Lemma 3
Using the law of iterated expectations we have

$$c(t,T) = \mathrm{E}^{\mathbb{Q}}\left[\max\left(IDI_t - K\mathrm{e}^{-H(t,T)}, 0\right)\Big|\, \mathcal{F}_t\right] =$$

$$\mathrm{E}^{\mathbb{Q}}\left[\mathrm{E}^{\mathbb{Q}}\left[\max\left(IDI_t - K\mathrm{e}^{-H(t,T)}, 0\right)\Big|\, \mathcal{G}_{t,T}\right]\Big|\, \mathcal{F}_t\right].$$

Then, the proof consists of a simple calculation of the ordinary integral $\mathrm{E}^{\mathbb{Q}}\left[\max\left(IDI_t - K\mathrm{e}^{-H}, 0\right)\middle|\, \mathcal{G}_{t,T}\right]$.

$$c\,(t,T) = \mathrm{E}^{\mathbb{Q}}\left[\max\left(IDI_t - K\mathrm{e}^{-H}, 0\right)\middle|\, \mathcal{G}_{t,T}\right] =$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi V\,(t,T)}} \max\left(IDI_t - K\mathrm{e}^{-h}, 0\right) \mathrm{e}^{-\frac{(h-M(t,T))^2}{2V(t,T)}}\,\mathrm{d}h =$$

$$= \int_{\log(K/IDI_t)}^{\infty} \frac{1}{\sqrt{2\pi V\,(t,T)}} \left(IDI_t - K\mathrm{e}^{-h}\right) \mathrm{e}^{-\frac{(h-M(t,T))^2}{2V(t,T)}}\,\mathrm{d}h.$$

Making the substitution $z = \frac{h-M(t,T)}{\sqrt{V(t,T)}}$ we have:

$$c\,(t,T) = \int_{-d}^{\infty} \frac{1}{\sqrt{2\pi}} \left(IDI_t - K\mathrm{e}^{-z\sqrt{V(t,T)}-M(t,T)}\right) \mathrm{e}^{-\frac{1}{2}z^2}\mathrm{d}z =$$

$$= IDT_t \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}z^2}\mathrm{d}z - K\int_{-d}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-z\sqrt{V(t,T)}-M(t,T)-\frac{1}{2}z^2}\mathrm{d}z =$$

$$= IDI_t \Phi\,(d) - K\mathrm{e}^{-M(t,T)+\frac{V(t,T)}{2}} \int_{-d}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\left(z+\sqrt{V(t,T)}\right)^2}\mathrm{d}z.$$

where d is given by Equation 39. Making a new substitution $w = z + \sqrt{V\,(t,T)}$ and using Lemma 2 results in Equation 38. □

Proof of Lemma 4

$$\mathrm{E}^{\mathbb{Q}}\left(V(t,T)|\mathcal{F}_t\right) =$$

$$V_{YZ}(t,T) + \frac{1}{\kappa^2}\int_t^T \left(1 - \mathrm{e}^{-\kappa(T-u)}\right)^2 \mathrm{E}^{\mathbb{Q}}\left(v_u|\mathcal{F}_t\right)\mathrm{d}u =$$

$$V_{YZ}(t,T) + \frac{1}{\kappa^2}\int_t^T \left(1 - \mathrm{e}^{-\kappa(T-u)}\right)^2 \left(v_t\mathrm{e}^{-\beta(u-t)} + \frac{\alpha}{\beta}\left(1 - \mathrm{e}^{-\beta(u-t)}\right)\right)\mathrm{d}u,$$

where in last step we have used the property of the mean of a CIR process (see Brigo and Mercurio (2001)). Expanding the terms in the right side and calculating the ordinary integrals give the desired result. □

Appendix B - Conditional Variance in Affine Models

Bond yields conditional variances under general affine models were provided in Almeida et al. (2006), and Jacobs and Karoui (2006). We specialize their results to the model proposed here.

The covariance matrix $var_t(E_{t+s})$ can be calculated by the following algorithm[33]:

1. Let $D \in \mathbb{R}^{25\times 5}$ be the matrix such that $D_{ij} = 1$ if $i = 5(j-1)+j$ and 0 otherwise.

2. Consider the matrices

$$\kappa_e = \begin{bmatrix} \kappa & 0 & 0 & -\kappa & 0 \\ 0 & \eta_y & 0 & 0 & 0 \\ 0 & 0 & \eta_z & 0 & 0 \\ 0 & 0 & 0 & 2\kappa & -1/\kappa \\ 0 & 0 & 0 & 0 & \beta \end{bmatrix},$$

$$\theta_e = \begin{bmatrix} \dfrac{\gamma + \frac{\alpha}{\beta\kappa}}{2\kappa} & 0 & 0 & \dfrac{\gamma + \frac{\alpha}{\beta\kappa}}{2\kappa} & \dfrac{\alpha}{\beta} \end{bmatrix}',$$

$$\Sigma_e = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_Y & 0 & 0 & 0 \\ 0 & \rho_{YZ} & \sigma_Z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\alpha_e = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \end{bmatrix}',$$

[33]For more details on this computation see Fackler (2000).

and

$$\beta_e = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta^2 \end{bmatrix}.$$

3. Next, compute the 30×30 matrix

$$K = \begin{bmatrix} \kappa_e & 0 \\ -(\Sigma \otimes \Sigma) D\beta_e & \kappa_e \otimes I_5 + I_5 \otimes \kappa_e \end{bmatrix}$$

   and the vector $\Theta \in \mathbb{R}^{30}$ defined by

$$\Theta = \begin{bmatrix} \kappa_e \theta_e \\ (\Sigma \otimes \Sigma) D\alpha_e \end{bmatrix},$$

   where $\otimes$ stands for the Kronecker product operator (see Horn and Johnson (1994)) and I_n is the $n \times n$ identity matrix.

4. For a fixed time to maturity s, compute the vector $Q_0 \in \mathbb{R}^{30}$ and the matrix $Q_1 \in \mathbb{R}^{30\times 5}$ defined as

$$Q_0 = \left(I_{30} - \mathrm{e}^{-Ks}\right) K^{-1}\Theta$$

   and

$$Q_1 = \mathrm{e}^{-Ks} \begin{bmatrix} I_5 \\ 0 \end{bmatrix}$$

5. Then

$$vec\left(var_t\left(E_{t+s}\right)\right) = V_0 + V_1 E_t$$

   where vec denotes the vectorized representation of a matrix, $V_0 \in \mathbb{R}^{25}$ is the vector composed by the last 25 elements of Q_0 and $V_1 \in \mathbb{R}^{25\times 5}$ is the matrix formed by the last 25 lines of Q_1.

Parameter	Value	Standard Error	ratio $\frac{\text{abs(Value)}}{\text{Std Err.}}$
κ	5.35e-6	1.23e-7	**43.61**
γ	17.41	0.6665	**26.11**
α	0.0112	0.0007	**16.80**
β	52.68	2.89	**18.17**
δ	0.1495	0.0037	**40.41**
η_Y	83.48	42.73	1.95
η_Z	3.1242	0.0936	**33.36**
σ_Y	0.0817	0.0406	**2.03**
ρ_{YZ}	-0.0010	0.0003	**3.36**
σ_Z	0.0200	0.0007	**27.87**
λ_0^X	0	-	-
λ_1^X	-5.0005	2.3819	**2.10**
λ_0^Y	-0.2371	0.1312	1.805
λ_1^Y	72.99	42.71	1.708
λ_0^Z	0.1011	0.0066	**14.69**
λ_1^Z	1.0001	0.2112	**4.73**
λ^{YZ}	0	-	-
λ_0^v	0.0182	0.0022	**8.08**
λ_1^v	50.0046	2.91	**17.19**
ϕ_0	0.15	-	-

Table 1: Parameters and Standard Errors.

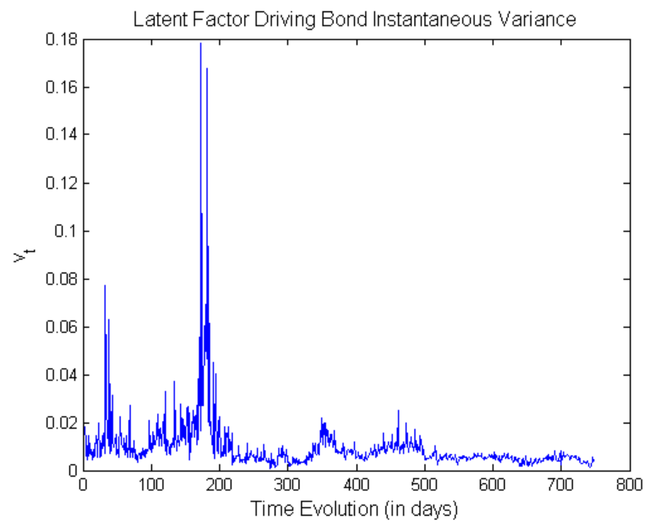

Figure 1: Instantaneous Variance of the Term Structure Level Factor.

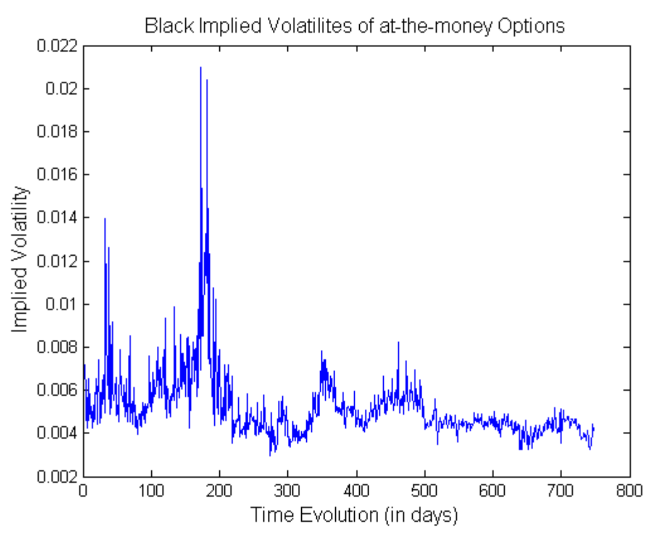

Figure 2: Black Implied Volatility of the Fixed-Maturity At-the-money Option.

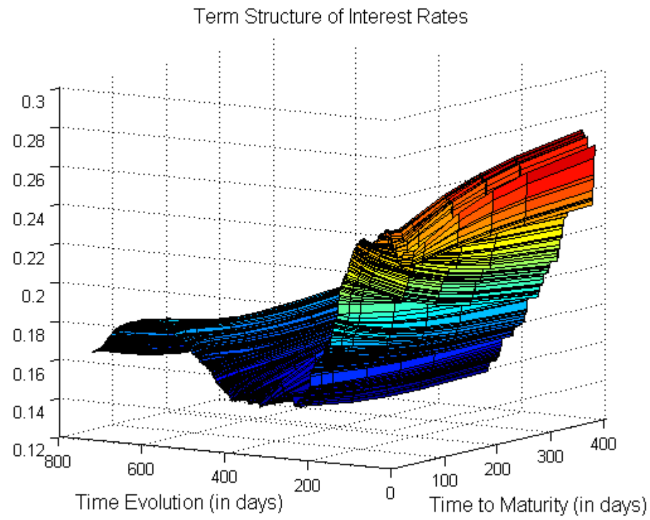

Figure 3: Historical Term Structure Evolution.

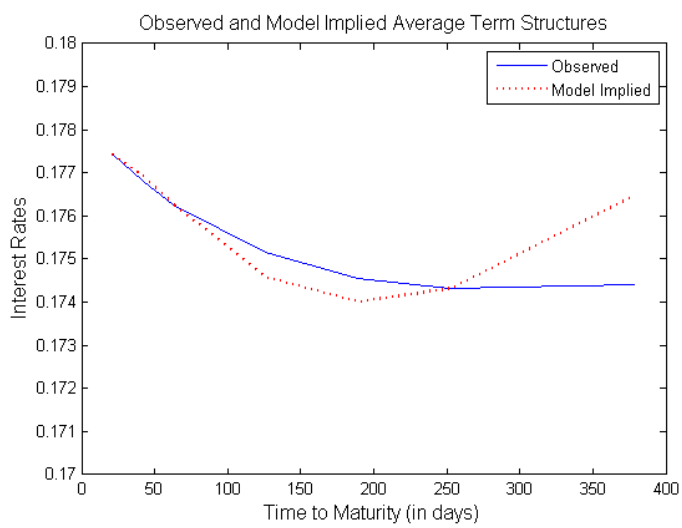

Figure 4: Observed and Model Implied Cross Section of Yields Averaged Across Sample.

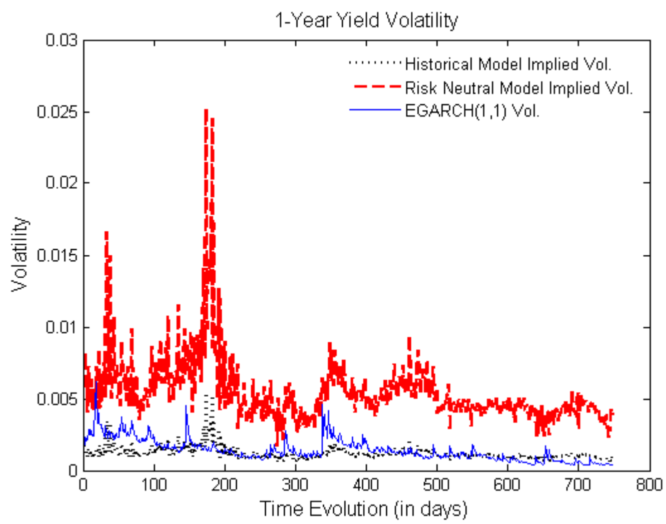

Figure 5: Volatility of the one-year Bond Yield: EGARCH Benchmark and Model Implied (Historical and Risk Neutral)

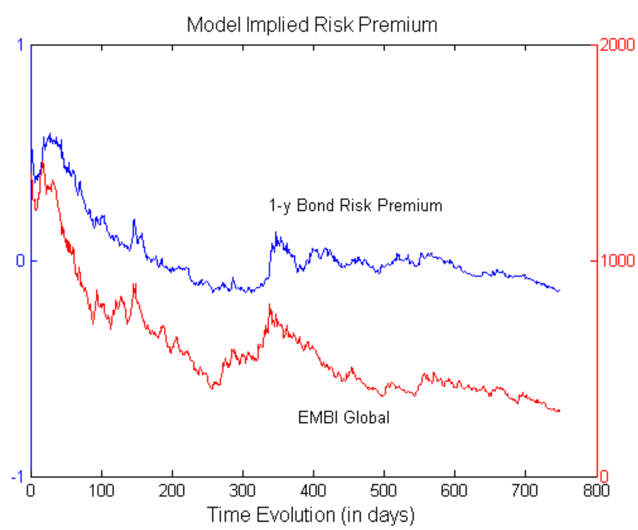

Figure 6: Bond Risk Premium and the EMBI-Global Index.

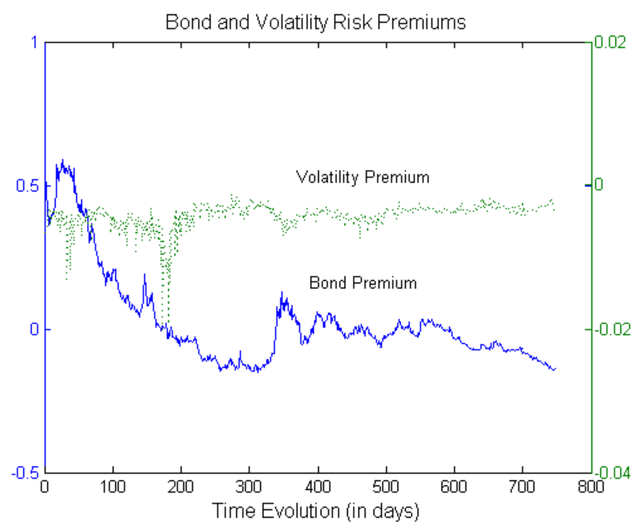

Figure 7: Bond and Volatility Risk Premiums.

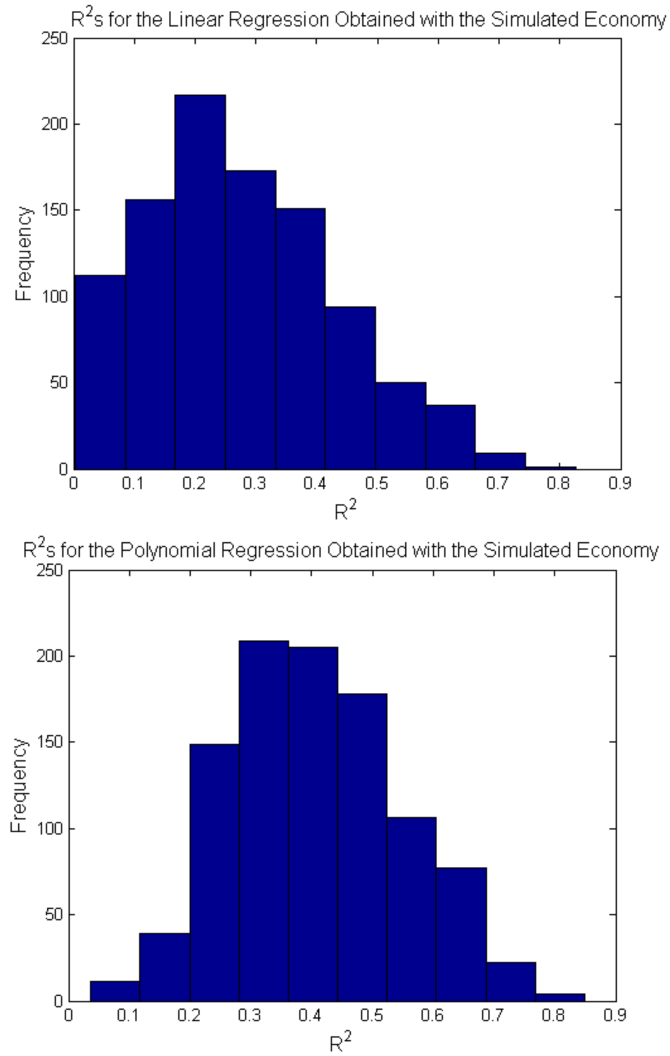

Figure 8: Testing the Transmission of Information in a Simulated Economy Under the USV Model.