

Pricing and Modeling Credit Derivatives*

Muzaffer Akat**
Caio Almeida***
George Papanicolaou****

Abstract

The market involving credit derivatives has become increasingly popular and extremely liquid in the most recent years. The pricing of such instruments offers a myriad of new challenges to the research community as the dimension of credit risk should be explicitly taken into account by a quantitative model. In this paper, we describe a doubly stochastic model with the purpose of pricing and hedging derivatives on securities subject to default risk. The default event is modeled by the first jump of a counting process N_t , doubly stochastic with respect to the Brownian filtration which drives the uncertainty of the level of the underlying state process conditional on no-default event. By assuming a condition slightly stronger than no arbitrage, *i.e.*, that there is *no free lunch with vanishing risk* (NFLVR) from Delbaen and Schachermayer (1994), we provide all the possible equivalent martingale measures under this setting. In order to illustrate the method, two simple examples are presented: the pricing of defaultable stocks, and a framework to price multi-name credit derivatives such as basket defaults.

Keywords: Intensity Models, Credit Derivatives, Counting Processes, No-Arbitrage Restrictions.

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**Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213-3890. E-mail: akat@andrew.cmu.edu

***Graduate School of Economics, Getulio Vargas Foundation, Praia de Botafogo, 190, 22250-040, Rio de Janeiro, RJ, Brazil. E-mail: calmeida@fgv.br

****Department of Mathematics, Stanford University, Stanford, CA 94305-2125.
E-mail: papanico@math.stanford.edu

1. Introduction

The market for credit derivatives has become increasingly popular and extremely liquid in the most recent years. Credit risk is basically everywhere, in swaps, corporate bonds, collateralized debt obligation (CDO), basket default instruments, sovereign bonds, etc. The necessity of explicitly considering credit risk by making use of quantitative modeling techniques is evident. For instance, Duffie and Singleton (1999) proposed a reduced form model for term structures of defaultable bonds, which is extended by Collin Dufresne et al. (2004), while Hull and White (2004) propose an implementation of Merton's (1974) seminal credit risk model estimated from the implied volatilities of options on the companies' equities, which outperform the original model. Hundreds of papers on credit risk modeling, which can be found on websites like "defaultrisk.com" and "gloriamundi.org", suggest this topic as an important issue under consideration.

Credit risk models are usually obtained by one of two concurrent techniques: structural models or reduced form models. In structural models, whose first representative is Merton's model (1974), a default is triggered when the process representing the assets of the firm falls below a certain barrier value. In reduced form models, whose first formal description appears in Duffie and Singleton (1999),¹ the default event is modeled by the first jump of a counting process N_t . Duffie and Singleton propose that N_t should be a doubly stochastic process, which means, a conditionally Poisson process, with the uncertainty driving the intensity of the process coming from a sigma-algebra which does not contain information regarding the jumps of this process. In their paper however, all the calculations are obtained under the risk neutral measure, with no allusion to the physical measure, under which we actually observe the dynamics of assets.

In this paper, we describe a doubly stochastic model with the purpose of pricing and hedging derivatives on securities subject to default risk. The default event is modeled by the first jump of a counting process N_t , doubly stochastic with respect to a Brownian filtration which drives the uncertainty of the volatility process, and of the security price conditional on no-default event. By assuming the condition first proposed by Delbaen and Scharchermayer (1994), which is slightly stronger than no arbitrage, namely *no free lunch with vanishing risk*,² we provide all the possible equivalent martingale measures under this setting. This allows

¹Others have used special cases of this approach before. For instance, Pye (1974) proposes a discrete time precursor version of this model where interest rates, and default intensities are deterministic.

²An arbitrage is a trading strategy that costs nothing in first place, and in future states of the nature, it generates positive value with a positive probability. The no-arbitrage condition rules out the set that contains this type of strategies, while the NFLVR condition rules out all strategies that appear in the closure of this set. We refer the reader to Delbaen and Scharchermayer (1994, 1998) for a detailed description of this property and examples. Duffie (2001) also presents a discussion about a corresponding property that he denominated absence of approximate arbitrages.

the implementation of dynamic credit risk models where investors can charge risk premia which take into account the probability that default events happen. Our main contribution is a clear description of the dynamics of credit default securities under both measures.

Other works have dealt with the problem of characterizing the price of risk for defaultable instruments. We present and discuss some of those references which are close in spirit to our paper. Giesecke and Goldberg (2005) extend the reduced-form intensity model by Duffie and Singleton (1999) to a structural form, and derive general credit derivatives pricing formulas and credit risk premium for models whose jump components do not admit intensity processes.³ In contrast, we analyze models whose jump components admit an intensity process, which for practical purposes will be extremely useful⁴ and that arguably deserve their own separate analysis. El Karoui and Martellini (2001) derive the change of measure from physical to equivalent martingale measures in the context of intensity-based models, where intensities are driven by Brownian motions correlated to the basic stock price. Despite the similarity to our work, allowing for this correlation between Brownian motions, they do not price the independent⁵ fraction of the diffusive risk that affects their intensity processes, while in this work we do. Allowing a more general price of risk as it appears here is important to capture the correlation between asset price changes and conditional probabilities of default changes, a demanded ability when pricing credit risk products with multiple assets. Kusuoka (1999) discusses the danger of assuming that the doubly stochastic property of a jump-diffusion process will be maintained once a change of measure is performed, with overall mathematical arguments that nest our analysis. On the other hand, we provide and analyze a concrete reduced-form financial model which might be directly applied in trading desks, as opposed to his more general abstract set of calculations.⁶

In addition to the carefully derived theoretical results presented in this paper, we also provide two examples of applications using our methodology: The pricing of defaultable equities (see Bielecki et al. (2004) or Das and Sundaram (2004)), and the pricing of multi-name products, such as basket defaults and CDOs (see Duffie and Singleton (2003)). For the multi-name example, we propose modeling the intensity of the counting process that drives default as a hybrid between a mean reverting state vector capturing specific firm risks and a function of a market index that would capture common factor risks. We show that the inclusion of this common factor improves the ability of the model in capturing correlation between default times of different companies, and might be able to attenuate part of the

³These jump processes present trends that are not absolutely continuous.

⁴See, for instance, Das et al. (2007) for a recent application of doubly stochastic processes to predict multiperiod corporate defaults.

⁵Independent from the stock diffusive risk.

⁶It is important to remark that Kusuoka (1999) indeed provides a short example, but based on the structural model of incomplete accounting information by Duffie and Lando (2001).

failings of the doubly stochasticity assumption in completely capturing correlated corporate defaults (see Duffie et al. (2007)).

The paper is organized as follows. Section 2 describes the basic framework, provides the theoretical results on changes of measures under our model, and proposes a first simple example of a defaultable stock. Section 3 considers how well the model is able to capture correlation of default times when dealing with multi-name securities. Section 4 concludes the article. The Appendix presents Girsanov's theorem for counting processes and an extension of the model that includes stochastic volatility for the underlying asset.

2. Pricing Defaultable Securities

Let us fix a probability space (Ω, \mathcal{F}, P) and the σ - algebra $\mathcal{F}_t = \sigma(W_t^S, Z_t^\lambda)$ where W_t^S, Z_t^λ are independent standard Brownian motions. Let us also introduce the σ - algebra $\mathcal{G}_t = \sigma(\mathcal{F}_t \vee N_t)$ where N_t is a nonexplosive doubly stochastic (with respect to \mathcal{F}_t) counting process with intensity λ_t , *i.e.*

- i. λ_t is \mathcal{F}_t predictable and $\int_0^t \lambda_s ds < \infty$ a.s.
- ii. $N_t - \int_0^t \lambda_s ds$ is a \mathcal{G}_t local martingale
- iii. $P\{N_s - N_t = k | \mathcal{G}_t \vee \mathcal{F}_s\} = \frac{e^{-\int_t^s \lambda_u du} (\int_t^s \lambda_u du)^k}{k!}$

In this section, we will price derivatives on a defaultable security, where the price of the security is modeled as a geometric Brownian motion, and the default event is modeled by the stopping time τ , the first jump of the counting process N_t . We assume that the short-term interest rate process is constant and equal to r .⁷ The default intensity process itself is modeled by an OU process. We also introduce correlation between the security price and the intensity Brownian motions allowing the changes in prices to influence the likelihood of default.

$$\begin{cases} dS_t &= \mu S_t dt + \sigma S_t dW_t^S \\ d\lambda_t &= a(b - \lambda_t)dt + \beta_\lambda dW_t^\lambda \end{cases} \quad dW_t^\lambda = \rho_\lambda dW_t^S + \sqrt{1 - \rho_\lambda^2} dZ_t^\lambda \quad (1)$$

where S is the stock price, σ is the volatility, λ is the instantaneous probability of default of the underlying asset.

Clearly, the way the problem is set up gives rise to an incomplete market model in the sense that there exist derivatives that can not be hedged by a portfolio of the basic securities. Assumption of *no free lunch with vanishing risk* guarantees the existence of a set of equivalent martingale measures.⁸ In this setting, an EMM P^*

⁷This implies that the money market account, the usual instrument adopted for deflation, will be $B_t = e^{rt}$.

⁸Non-empty, non-unitary.

is a probability measure equivalent to P , under which the discounted price of the defaultable security,⁹ $e^{-rt}S_t1_{\{\tau>t\}}$ is a \mathcal{G}_t -sigma-martingale.¹⁰ Note that although the equity price S_t does not jump, the defaultable equity price $e^{-rt}S_t1_{\{\tau>t\}}$ jumps due to the indication function which directly depends on the counting process N . At this point, we look for all possible EMM's P^* that allow us to write the price of a defaultable object as an expectation in terms of the intensity of the counting process N_t under P^* .¹¹ Let the set of all such measures be \mathcal{S} .

In order to characterize all EMM's in the set \mathcal{S} , we make use of the two versions of the Girsanov's theorem, where one is for changes in the Brownian filtration and one for the changes in the intensity process λ_t . In order to construct our argument, we state the standard version of Girsanov's theorem for a d -dimensional Brownian filtration¹² and also the Girsanov's theorem version for counting processes.¹³

2.1 Girsanov's theorem (G1):

Given $\theta \in (\mathcal{L}^2)^d$, assume that $\xi_t^\theta = e^{-\int_0^t \theta_s dW_s - \frac{1}{2} \int_0^t \theta_s \cdot \theta_s ds}$ is a martingale (Novikov's condition is sufficient.) Then a standard Brownian motion W^θ is defined by

$$W_t^\theta = W_t + \int_0^t \theta_s ds, \quad 0 \leq t \leq T$$

Moreover, W^θ has the martingale representation theorem under the new measure P^* where $\frac{dP^*}{dP} = \xi_T^\theta$. Hence, any P^* martingale can be represented as

$$M_t = M_0 + \int_0^t \phi_s dW_s^\theta, \quad t \leq T$$

for some $\phi \in (\mathcal{L}^2)^d$

⁹Defaultable equity derivatives have been also studied in Bielecki et al. (2004) and Das and Sundaram (2004).

¹⁰A sigma-martingale is a generalization of a local martingale. A process X is a sigma-martingale if it is a semimartingale and there exists a martingale M and an M -integrable predictable process η such that $X = \eta M$. See Delbaen and Scharchermayer (1998) for more details and examples.

¹¹More general versions, useful, for instance, in markets with multiple defaults, would allow the intensity to depend not only on a Brownian filtration, but also on default events.

¹²For the proof, see Karatzas and Shreve (1991).

¹³For the proof, see Appendix.

2.2 Girsanov’s theorem for counting processes (G2):

Suppose N_t is a nonexplosive counting process with intensity λ_t , and ϕ is a strictly positive predictable process such that, for some fixed T , $\int_0^T \phi_s \lambda_s ds < \infty$ almost surely. Then,

$$\xi_t^\phi = e^{\int_0^t (1-\phi_s)\lambda_s ds} \prod_{\{i:\tau(i)\leq t\}} \phi_{\tau(i)}$$

is a well defined local martingale where $\tau(i)$ is the i^{th} jump time of N_t . In addition, if ξ_t^ϕ is a martingale (bounded ϕ suffices), then an equivalent martingale measure P^* is defined by $\frac{dP^*}{dP} = \xi_T^\phi$. Under this new martingale measure, N_t is still a nonexplosive counting process with intensity $\lambda_t \phi_t$.

Now, suppose the counting process N_t is doubly stochastic with respect to \mathcal{F}_t under measure P^* ,¹⁴ say with an intensity λ^* . Then one can show that:

$$\begin{aligned} E^*\{1_{\{\tau>s\}}|\mathcal{G}_t \vee \mathcal{F}_s\} &= E^*\{1_{\{\tau>t\}}1_{\{N_s-N_t=0\}}|\mathcal{G}_t \vee \mathcal{F}_s\} & (2) \\ &= 1_{\{\tau>t\}}E^*\{1_{\{N_s-N_t=0\}}|\mathcal{G}_t \vee \mathcal{F}_s\} & (3) \\ &= 1_{\{\tau>t\}}e^{-\int_t^s \lambda_u^* du} & (4) \end{aligned}$$

If we go from P^* to a measure P^{**} making use of G1, then one can prove that there is an equivalence between the discounted defaultable price being a \mathcal{G}_t P^{**} -martingale and the process $e^{\int_0^t (r+\lambda_u^*)du} S_t$ being a \mathcal{G}_t P^{**} -martingale as follows

$$\begin{aligned} E^{**}\{e^{-rs}S_s1_{\{\tau>s\}}|\mathcal{G}_t\} &= E^*\left\{\frac{\xi_s}{\xi_t}e^{-rs}S_s1_{\{\tau>s\}}|\mathcal{G}_t\right\} \\ &= E^*\left\{E^*\left\{\frac{\xi_s}{\xi_t}e^{-rs}S_s1_{\{\tau>s\}}|\mathcal{G}_t\right\}|\mathcal{G}_t \vee \mathcal{F}_s\right\} \\ &= E^*\left\{E^*\left\{\frac{\xi_s}{\xi_t}e^{-rs}S_s1_{\{\tau>s\}}|\mathcal{G}_t \vee \mathcal{F}_s\right\}|\mathcal{G}_t\right\} \\ &= E^*\left\{\frac{\xi_s}{\xi_t}e^{-rs}S_sE^*\{1_{\{\tau>s\}}|\mathcal{G}_t \vee \mathcal{F}_s\}|\mathcal{G}_t\right\} \\ &= 1_{\{\tau>t\}}e^{\int_0^t \lambda_u du}E^*\left\{\frac{\xi_s}{\xi_t}e^{-\int_0^s (r+\lambda_u^*)du}S_s|\mathcal{G}_t\right\} \\ &= 1_{\{\tau>t\}}e^{\int_0^t \lambda_u du}E^{**}\left\{e^{-\int_0^s (r+\lambda_u^*)du}S_s|\mathcal{G}_t\right\} \end{aligned}$$

where we used Bayes rule and equation (2, 3, 4).

¹⁴In our context, we control the doubly stochasticity of N under P^* through function ϕ , avoiding the appearance of pathological examples like those provided by Kusuoka (1999).

How do we characterize the elements in \mathcal{S} ? The idea is to perform a two-step change of measure, where we first change the intensity of the counting process going from P to P^* , using a particular case of G2 where N_t counts up to 1.¹⁵ Then, we apply G1 by changing the measure from P^* to P^{**} . The Radon-Nikodym derivative from P to P^{**} is, by construction, the product of the two Radon-Nikodym derivatives from P to P^* and from P^* to P^{**} . According to theorems G1 and G2

$$\frac{dP^*}{dP} = e^{\int_0^t (1-\phi_s)\lambda_s ds} (\mathbf{1}_{\{\tau>t\}} + \phi(\tau)\mathbf{1}_{\{\tau\leq t\}}) \quad \text{and} \quad \frac{dP^{**}}{dP^*} = e^{-\int_0^t \theta_s dW_s - \frac{1}{2} \int_0^t \theta_s^2 ds}$$

In general, ϕ and θ are free parameters, but for simplicity we assume that ϕ is deterministic, and to guarantee that the process $e^{-rt}\mathbf{1}_{\{\tau>t\}}S_t$ is a \mathcal{G}_t martingale, we choose the θ as follows:

$$\theta_t = \left[\frac{\mu-r-\lambda_t^*}{\sigma} \delta_t \right]$$

where the parameter δ is free. The Radon-Nikodym derivative of the new measure becomes

$$\frac{dP^{**}}{dP} \Big|_{F_t} = e^{-\int_0^t \theta_1(u) dW_u^S - \frac{1}{2} \int_0^t \theta_1^2(u) du} e^{-\int_0^t \theta_2(u) dZ_u^\lambda - \frac{1}{2} \int_0^t \theta_2^2(u) du} e^{\int_0^t (1-\phi_u)\lambda_u du} (\mathbf{1}_{\{\tau>t\}} + \phi(\tau)\mathbf{1}_{\{\tau\leq t\}})$$

where $\theta_1 = \frac{\mu-r-\lambda_t^*}{f(Y_t)}$, $\theta_2 = \delta_t$ and system (1) becomes:

$$\begin{cases} dS_t &= (r + \lambda_t^*)S_t dt + \sigma S_t dW_t^{*S} \\ d\lambda_t^* &= \left[\left(a - \frac{\phi_t'}{\phi_t} \right) \left(\frac{ab\phi_t}{a - \frac{\phi_t'}{\phi_t}} - \lambda_t^* \right) - \beta_\lambda \phi_t \left(\rho_\lambda \frac{\mu-r-\lambda_t^*}{f(Y_t)} \right. \right. \\ &\quad \left. \left. + \delta_t \sqrt{1 - \rho_\lambda^2} \right) \right] dt + \beta_\lambda \phi_t dW_t^{*\lambda} \end{cases} \quad (5)$$

Let us introduce the two-dimensional vector process X_t as follows: $X_t = [S_t \lambda_t^*]$. Then the process followed by X_t can be written as:

$$dX_t = \begin{bmatrix} (a - \frac{\phi_t'}{\phi_t}) \left(\frac{ab\phi_t}{a - \frac{\phi_t'}{\phi_t}} - \lambda_t^* \right) - \beta_\lambda \phi_t \left(\rho_\lambda \frac{\mu-r-\lambda_t^*}{\sigma} + \delta_t \sqrt{1 - \rho_\lambda^2} \right) & r + \lambda_t^* \\ \sigma S_t & 0 \end{bmatrix} dt + \begin{bmatrix} \phi_t \beta_\lambda \rho_\lambda & \phi_t \sqrt{1 - \rho_\lambda^2} \beta_\lambda \end{bmatrix} \begin{bmatrix} dW_t^{*S} \\ dZ_t^{*\lambda} \end{bmatrix}$$

¹⁵See Kusuoka (1999) for examples where a change of measure like the one proposed in this step destroys the doubly stochasticity of the jump process N_t .

directly implying that the Feynman-Kac PDE for the function $P(x, t) = E\{e^{-\int_0^t (r+\lambda_u^*) du} g(X_t)\}$, where $g(X_t) = h(S_t)$, is

$$P_t + (r + \lambda_t^*)P_S + \left[\left(a - \frac{\phi_t'}{\phi_t}\right)\left(\frac{ab\phi_t}{a - \frac{\phi_t'}{\phi_t}} - \lambda_t^*\right) - \beta\lambda\phi_t\left(\rho\lambda\frac{\mu - r - \lambda_t^*}{\sigma} + \delta_t\sqrt{1 - \rho_\lambda^2}\right)\right] \\ \frac{1}{2}\sigma_t^2 S_t^2 P_{SS} + \frac{1}{2}\phi_t^2 \beta_\lambda^2 P_{\lambda^* \lambda^*} + \sigma S_t \phi_t \rho_\lambda \beta_\lambda P_{S\lambda^*}$$

with the boundary condition $P(X_T, T) = h(S_T)$.

2.3 From actual to risk-neutral default intensities: The market price of credit risk role

In general, the pricing of derivative securities will be performed with the use of a conditional expectation under a risk-neutral measure, under which the jump process has a certain risk-neutral intensity λ^* . However, as stressed by Duffie and Singleton (2003) in section 5.6, in many cases, information about the intensities under both measures might be useful for pricing. For instance, they give the example of how difficult it would be to extract risk-neutral default probabilities from the prices of a newly issued security,¹⁶ and suggest using historical data about actual default probabilities for the firm. Another example that we add here is the pricing of a credit derivative which pays one dollar if the actual conditional probability of default exceeds a certain barrier value. In this case, the pricing is done under the risk-neutral measure, which implies the necessity of knowing λ^* for discounting purposes, but we also need the actual intensity λ to set up the derivative cash flow. In the present paper, the link between those two intensities is given by processes ϕ and δ , and they genuinely define the market price of credit risk.¹⁷ While process ϕ has direct effect on λ (it multiplies λ), process δ has an indirect effect coming from the change in the Brownian motions when we change measure. This Brownian motion change caused by δ affects the dynamics of λ^* and, as shown by Pan and Singleton (2006), might have an important role when pricing credit derivatives.¹⁸ We finally conclude this short discussion by noting that the particular setting provided in this paper is considerably flexible for the price of credit risk since it will allow in practical problems both sources of uncertainty of default probabilities (ϕ and δ) to be combined in an estimation procedure.

¹⁶As those derivative prices are formed by investor's expectations about probabilities of default, they will take a while to become stable.

¹⁷That will mean how much investors are willing to pay for uncertainty in the probabilities of default.

¹⁸In their particular application, δ is an affine function of the underlying factors that are driven by Brownian motions.

2.4 First example: Poisson process with constant intensity

Let S_t be a defaultable security which follows a geometric Brownian motion with constant mean and volatility parameters. The counting process modeling default events has a constant intensity λ , *i.e.*

- i. $dS_t = \mu S_t dt + \sigma S_t dW_t^S$
- ii. $N_t \sim \text{Poisson}(\lambda t)$

Define $\mathcal{F}_t = \sigma\{W_t^S\}$ and $\mathcal{G}_t = \sigma\{W_t^S, N_t\}$. We are interested in pricing a derivative based on S_t with a maturity date T . We would like to find an equivalent measure P^{**} under which the process $Y_t \equiv e^{-rt} S_t 1_{\{\tau > t\}}$ is a \mathcal{G}_t martingale. In the spirit of the argument for the general change of measure which appears in the previous section, we can change the measure using two steps. First, Girsanov's theorem for counting processes guarantees that if we take ϕ constant and perform the change of measure $\frac{dP^*}{dP} = e^{\int_0^T (1-\phi)\lambda du} (1_{\{\tau > T\}} + \phi 1_{\{\tau \leq T\}})$, N_t is a Poisson process under P^* with intensity $\lambda^* = \phi\lambda$.

Now, we want to make a second change of measure from P^* to P^{**} by just changing the Brownian filtration in a way that $Y_t \equiv e^{-rt} S_t 1_{\{\tau > t\}}$ is a \mathcal{G}_t martingale under P^{**} . We know that this change will not affect the characteristics of the process N_t which will consequently be Poisson under P^{**} with intensity $\lambda^* = \phi\lambda$. According to the previous section, the general change of measure is

$$\frac{dP^{**}}{dP} = e^{-\int_0^T \frac{\mu-r-\lambda^*}{\sigma} dW_u^S - \frac{1}{2} \int_0^T \left(\frac{\mu-r-\lambda^*}{\sigma}\right)^2 du} e^{\int_0^T (1-\phi)\lambda du} (1_{\{\tau > T\}} + \phi 1_{\{\tau \leq T\}})$$

Finally, it is interesting to note that, under this simple model, the problem of pricing a derivative on the defaultable stock boils down to pricing a derivative on the non-defaultable stock with default adjusted parameters that represent a spread in interest rates:

$$dS_t = (r + \lambda_t^*) S_t dt + \sigma S_t dW_t^{*S}$$

2.5 Second example: Poisson process with random intensity but constant risk premium

Define the following σ -algebras $\mathcal{F}_t = \sigma\{W_t^S, Z_t^\lambda\}$ and $\mathcal{G}_t = \sigma\{W_t^S, Z_t^\lambda, N_t\}$, where W_t^S and Z_t^λ are two independent Brownian motions, and N_t is a counting process, doubly stochastic with respect to \mathcal{F}_t . Let S_t be a defaultable stock which follows a geometric Brownian motion with constant mean and volatility parameters. The default process is modeled by the first jump of the counting process N_t . The stochastic intensity λ follows an OU process driven by a Brownian motion that is correlated with the Brownian motion which drives the dynamics of the stock price. The model is described by the following equations:

- i. $dS_t = \mu S_t dt + \sigma S_t dW_t^S$
- ii. $d\lambda_t = a(b - \lambda_t)dt + \beta_\lambda dW_t^\lambda, \quad dW_t^\lambda = \rho_\lambda dW_t^S + \sqrt{1 - \rho_\lambda^2} dZ_t^\lambda$
- iii. $N_s - N_t | \mathcal{G}_t \vee \mathcal{F}_s \sim \text{Poisson} (\int_t^s \lambda_u du)$

We would like to find a general measure P^{**} equivalent to P , under which the process $Y_t \equiv e^{-rt} S_t 1_{\{\tau > t\}}$ is a \mathcal{G}_t martingale. Using the same idea of the previous example, we first apply Girsanov's theorem for counting processes by taking ϕ constant and perform the change of measure $\frac{dP^*}{dP} = e^{\int_0^T (1-\phi)\lambda du} (1_{\{\tau \geq T\}} + \phi 1_{\{\tau \leq T\}})$. Again, we want to make a second change of measure from P^* to P^{**} by just changing the Brownian filtration in a way that $Y_t \equiv e^{-rt} S_t 1_{\{\tau > t\}}$ is a \mathcal{G}_t martingale under P^{**} . Then, the general change of measure is

$$\frac{dP^{**}}{dP} = e^{-\int_0^T \frac{\mu - r - \lambda_u^*}{\sigma} dW_u^S - \frac{1}{2} \int_0^T (\frac{\mu - r - \lambda_u^*}{\sigma})^2 du} e^{-\int_0^T \delta_u dW_u^\lambda - \frac{1}{2} \int_0^T \delta_u^2 du} e^{\int_0^T (1-\phi)\lambda_u^* du} (1_{\{\tau > T\}} + \phi 1_{\{\tau \leq T\}})$$

The equations for the security price and intensity processes under P^{**} are given by:

$$\begin{cases} dS_t &= (r + \lambda_t^*) S_t dt + \sigma_t S_t dW_t^{**S} \\ d\lambda_t^* &= [a(\phi b - \lambda_t^*) - \phi \beta_\lambda (\rho_\lambda \frac{\mu - r - \lambda_t^*}{\sigma} + \sqrt{1 - \rho_\lambda^2} \delta_t)] dt + \phi \beta_\lambda dW_t^{**\lambda} \end{cases} \quad (6)$$

Let us assume, for instance, that $\rho_\lambda = 0$ and that δ is an affine function of λ_t^* . In particular, $\delta_t = m\lambda_t^*$. Then the equation for the intensity process takes the form

$$d\lambda_t^* = (a + \beta_\lambda m) (\frac{\phi b a}{a + \beta_\lambda m} - \lambda_t^*) dt + \phi \beta_\lambda dW_t^{*\lambda}$$

Observe that, ϕ and δ have different effects on the intensity process. While ϕ affects the long-term mean of the process and also the volatility of the process, δ affects the long-term mean and the speed of mean reversion.

3. Modeling the Default Correlation in Multi-Name Products

In today's financial markets, there are lots of multi-name products whose pricing is critically dependent on the correlation of defaults of these different names. Basket default swaps, CDOs, CBOs are some examples. In this section, we try to develop a model to price this kind of products by paying particular attention to default correlations. We try to combine the capital structure models and reduced form models by modeling the default intensities of different names as both a function of the overall market and a function of its individual structure. Modeling the

effect of overall market is done through a proxy such as a big common index, *e.g.* S&P 500 and the effect of individual structure is like a surprise default.

A fair amount of this section is devoted to explaining the ability of the model in capturing the high levels of correlation between defaults of different names. Since there are various definitions for default correlation in the literature, we would like to clarify which definition we will use throughout the section. The default correlation of two names, say S_1 and S_2 , is defined with respect to their exact default times, say τ_1 and τ_2 as follows:

$$\left\{ \begin{array}{l} \rho = \frac{\text{cov}(\tau_1, \tau_2)}{\sqrt{\text{var}(\tau_1)\text{var}(\tau_2)}} \\ = \frac{E\{\tau_1\tau_2\} - E\{\tau_1\}E\{\tau_2\}}{\sqrt{\text{var}(\tau_1)\text{var}(\tau_2)}} \end{array} \right. \quad (7)$$

Hereafter, we simply regard this definition as the default correlation. This kind of default correlation is a much more general concept than that of the discrete default correlation based on a one period, *i.e.* the correlation between defaults of two different names occurred or not in a certain period. Certainly, if the joint distributions of default times are known, we could calculate quantities like $E\{1_{\tau_1 < T} 1_{\tau_2 < T}\}$ or $E\{1_{\tau_1 < T}\}$ and therefore calculate the discrete default correlation using the methodology described in Lucas (1995). However, even if we know the discrete correlation, we cannot calculate the default correlation in the above sense.

The straight forward intuition behind the setting is the following: When the overall market is not doing well, the default probability of each name tends to go up together, not necessarily at the same rate. Or there could be something happening not in the whole market, but in a specific sector which would bump up all the default probabilities of names in that sector. In addition, there could be also something happening within a firm which would only affect that particular firm, but not the others. So it is natural to assume that the default probabilities (intensities) have two different components, one for the overall market effect and one for the individual firm effect.

In the typical setting of the model, the proxy used to capture the overall market impact is modeled as a geometric Brownian motion. All the default intensities are modeled as product of a state process, which is an OU process with appropriate parameters and a positive function of the index level above. The Brownian motions that drive the dynamics of all the state processes that affect the intensities are correlated with each other. One can, in general, introduce the correlation between the Brownian motions of the index level and state processes, but we would rather capture that effect in the specific form we choose for the intensity processes.

Now suppose we try to price a product that depends on N different names. Then the *SDEs* that describe the event look as follows:

$$\begin{cases} dS_t &= \mu S_t dt + \sigma_t S_t dW_t^S \\ \lambda_t^i &= X_t^i g_i(S_t, K^i) \quad \text{for } i = 1, 2, \dots, N \\ dX_t^i &= a^i (b^i - X_t^i) dt + \beta_X^i dW_t^i \quad E\{dW_t^i dW_t^j\} = \rho_{ij}^X dt \end{cases} \quad (8)$$

where S is the common index level, σ is the volatility of the index level, λ^i is the instantaneous probability of default of name i . X is the OU state process whose each component affects each specific intensity process (capturing the firm specific risk), and the function g is some power function which blows up at a certain fixed boundary level K^i . Although one can keep the function g general for the rest of the section, we will assume that $g(S_t, K) = (\frac{S_0 - K}{S_t - K})^n$ where $S_0 - K$ is a normalization factor and S_0 is the initial value of the index in the period of interest.

Although there seems to be a lot of parameters in the general setting, as far as the correlation of default times are concerned, there are just a few key parameters. The most important one is the actual exponent in the function g . Clearly, a positive power corresponds to a positive correlation and a negative power corresponds to a negative correlation between the market and the default intensity of the individual name. If we assume the same type of g , *i.e.* the same power and the same boundary level for two different names and keep the other variables fixed, we observe that the correlation is almost a linear function of the square root of this power parameter (See Figure 1).¹⁹ The difference between the two pictures is the level of the volatility of the common index process. So the first one corresponds to a market with high volatility and the second to one with low volatility.

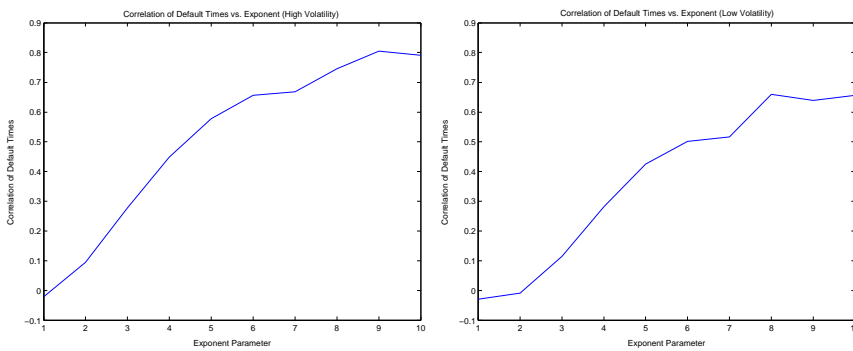


Figure 1

Correlation effect of the parameter in the exponent of function g under low volatile and high volatile environments

¹⁹This figure was obtained with 100,000 simulations performed with a fixed set of parameters that appears in Appendix 5.3. Parameters are similar to those adopted in previously published literature (see Duffie and Singleton (2003)).

As we can see from the figures above through this model, we get correlation between default times up to 90%. Here, the correlation is defined in the classical sense. In the market, the correlation between consecutive defaults, *i.e.* the defaults that happened within the same year, is also of interest. However, one gets just similar results for that definition of correlation, too.

At this point, we see that in the above pair of figures, although each of them contains almost straight lines, the slopes of those lines are different. This means that the level of correlation introduced by the specific form of function g creates different effects in different regimes in the market. Note that, besides the exponent and the volatility parameters, another very effective parameter is the explosion boundary K in function g . We call it explosion boundary because once the value of the common index gets close to this level, it increases all the intensities by an incredible amount and we get simultaneous defaults. Also, the more we are further away from this level, the smaller the intensities, *i.e.* when the market is doing well, all the default intensities tend to lower. And the closeness of this level to the index level is basically the sensitivity of the individual to the overall market. But in order to create a uniform effect of this exponent under different regimes of the process S_t , we also define this explosion boundary as a function of S_0 and σ and we let $K = S_0 - L * \sigma$. Then if we generate the first two pictures with this new definition of the boundary, we observe the same level of correlation effect under both regimes, as shown in Figure 2.

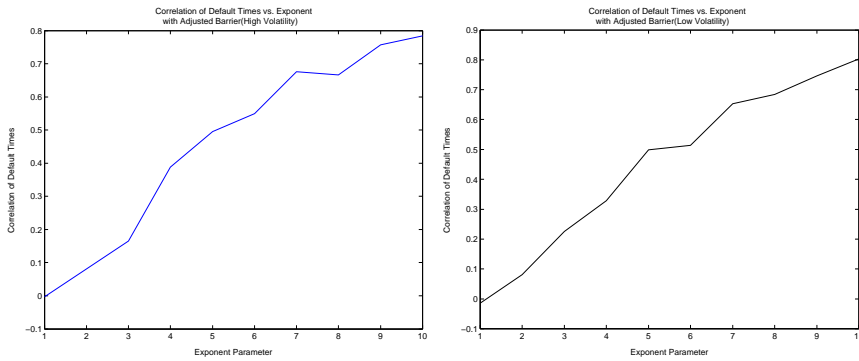


Figure 2

Correlation effect of the parameter in the exponent of function g without the effect of the volatility of the index process

Regarding the other set of parameters that could possibly affect the default time correlation, that is, the correlation parameters ρ_{ij}^X of the Brownian motions driving the state process X or the mean reverting level of the state processes,

we experimented the same phenomena under different sets of values of all those parameters: We observe that for all possible values of these parameters we get the same effect on the correlation of the default times. For instance, consider a first-to-default instrument. It is a contingent claim that pays off \$1 at the time where the first from a set of n names defaults. Figure 3 shows the effect of the correlation parameters between the state variables X_{iS} on the premium of a first-to-default insurance contract.

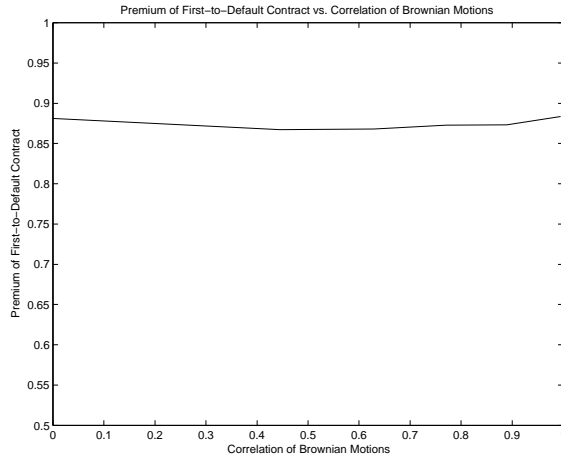


Figure 3

Correlation between Brownian motions *vs.* the premium of a first-to-default contract

As it is clear from the picture, there is almost no effect of the correlation parameter for Brownian motions on the price of the first-to-default instrument. This is an indication that the correlation parameter for the Brownian motions is not affecting the correlation of default times; otherwise, the price would be sensitive to it. Why? Let us provide a simple argument for why first-to-default price is sensitive to the correlation of default times. Suppose we are pricing a first-to-default derivative on a basket of n names. Well, if we have n perfectly uncorrelated default times, then the time of the first-to-default event will be the minimum of all default times (see Duffie (1998)), and having a large number of assets in the basket will certainly increase the premium of the first-to-default derivative. On the other hand, if the n times-to-default are all perfectly correlated, then having an insurance against the first-to-default or against any of the individual assets would be the same (if we assume that each name has the same default intensity). Therefore, the price of the contract should be much less in the case of the perfectly correlated case than in the uncorrelated one. In fact, one can use the very same

argument to convince oneself that the price of this contract is a decreasing function of the correlation of the default times. Hence, in our model, a decreasing function of the square root of the exponent parameter. Figure 4 shows the impact of the exponent on the price of the first-to-default contract. Note that the price is clearly a decreasing function of the exponent, except for a small region where the exponent assumes values around 0.4. This indicates that we can generate correlation on default times using our common factor captured by the market index.

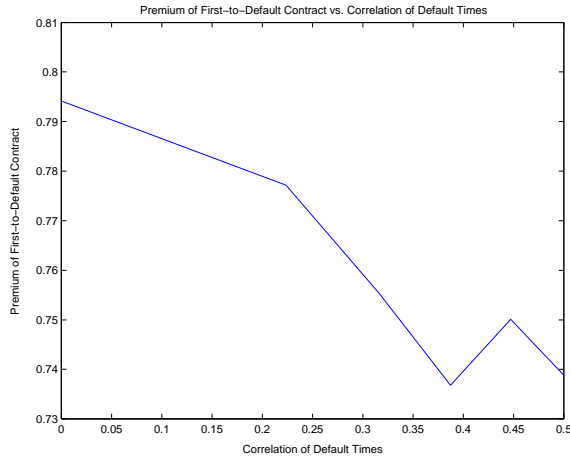


Figure 4
Correlation of default times *vs.* the premium of a first-to-default contract

3.1 A comparison with the Gaussian Copula model

We would like to compare the suggested model in the paper to a model that is commonly used in practice.²⁰ The market standard for modeling the credit default swap premia is the Gaussian copula model (see Li (2000)), and we compare the numbers produced by the two models in the example of a first-to-default CDS contract on N number of names.

In a first-to-default CDS, the protection buyer, that is, the counterparty that receives a payoff if one of the N reference entities defaults, pays a periodic premium to the protection seller. In return, the protection seller has to compensate the protection buyer in the case where the reference entity defaults prior to a pre-determined maturity time. The premia the protection buyer pays to the protection

²⁰We thank a referee for suggesting this comparison and for providing the CDS first-to-default prices of the Copula model.

seller are usually paid quarterly or semiannually until the maturity of the credit default swap contract, or until the time of the default event, whichever comes first.

To fix ideas, suppose that the CDS is written on a fixed notional amount F . In the case where one of the reference entities defaults before the maturity of the CDS contract, we assume that the protection seller compensates the protection buyer with a cash settlement. In particular, the protection seller will make a cash payment to the protection buyer equal to the recovery rate multiplied by the notional amount F . Here, we assume that in default, the buyer recovers $R\%$ of the asset face value and the protection seller provides the remaining proportion $1 - R\%$ of the face value to the protection buyer.

As an example, we calculate the premium of a first-to-default CDS premium using the following parameters with an expiration date of 5 years. We assumed the expected recovery rate, R , is 40% and the term structure of interest rates is assumed to be flat at 12%. If the structure of dependence is modeled by a Gaussian copula model using 500,000 Monte Carlo simulations, the default probabilities for the N entities generated by Poisson processes with constant default intensities .03, which have a constant pairwise correlation of .3, are reported in the following table.

To adopt this exercise in the view of our model, we keep all the parameters the same as the risk-free interest rate $r = 12\%$, the recovery rate $R = 40\%$ and the constant default intensity $x = .03$. However, we should keep in mind that our default intensity has two parts. Although we keep the exogenous part constant, it will still be stochastic due to its part with the function $g(S_t)$. Also, we do not have the correlation parameter of the copula model, but instead, we pick our parameter for the exponent in the function $n = 3.2$ which corresponds to 30% correlation of default times of two names, see Figure 1. We report the results in the following table.

Number of entities (N)	Premium by Gaussian copula	Premium by suggested model
3	100	104
5	224	172
10	516	356

Note that the larger the number of entities, the larger the difference between the copula model and the proposed model. One possible plausible explanation comes from the fact that those two models have very different modeling techniques, which might end up generating different correlation of default times, the main variable driving the CDS first-to-default prices. Note that the Gaussian copula generates correlation on the default barriers while the proposed model generates correlation on the default intensities. In any case, the smaller values obtained by the proposed model for the 5- and 10-name CDS indicate that it actually generates more correlation between default times of each name than the Gaussian copula model. We leave a more detailed numerical/empirical analysis of the proposed model *vis à vis* other available candidates for future work.

4. Conclusion

By adopting a doubly stochastic counting process to represent default events, we present theoretical results on the construction of equivalent martingale measures for defaultable securities. Examples are provided involving defaultable equities, and multi-name products. In order to price multi-name products, we propose the intensity of the counting process to be a combined function of both a mean reverting state vector representing firm-specific risk and a function of a common market index. We show that it is possible to generate correlation between default times of the different instruments that compose the basket by controlling one specific parameter of the common factor function (the exponent).

In the general theoretical model, two sources contribute to changes in the intensity λ of the counting process from the physical to the risk-neutral measure: the Brownian filtration which influences the dynamics of λ indirectly changing its behavior, and a possible deterministic shift ϕ that re-scales λ . These results might be applied to econometric studies of defaultable claims to determine the price of credit risk charged by investors, and in particular, if investors price credit risk sharply (significant ϕ) or more smoothly (significant change in the drift of Brownian motion driving λ).

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Appendix

Girsanov's Theorem for Counting Processes

Claim 1 : Suppose N_t is a nonexplosive counting process with intensity λ_t , and ϕ_t is a strictly positive predictable process such that, for some fixed T , $\int_0^T \phi_s \lambda_s ds < \infty$ almost surely. Then,

$$\xi_t^\phi = e^{\int_0^t (1-\phi_s)\lambda_s ds} \prod_{\{i:\tau(i)\leq t\}} \phi_{\tau(i)}$$

is a well defined local martingale where $\tau(i)$ is the i^{th} jump time of N_t .

Proof: Define

$$X_t = e^{\int_0^t (1-\phi_s)\lambda_s ds} \quad \text{and} \quad Y_t = \prod_{\{i:\tau(i)\leq t\}} \phi_{\tau(i)} \quad \text{and} \quad M_t = N_t - \lambda_t dt$$

then

- i. $\xi_t^\phi = X_t Y_t$;
- ii. M_t is a local martingale;
- iii. $dX_t = (1 - \phi_t)\lambda_t e^{\int_0^t (1-\phi_s)\lambda_s ds} dt = (1 - \phi_t)\lambda_t X_t dt$;
- iv. $dY_t = \left(\prod_{\{i:\tau(i)\leq t\}} \phi_{\tau(i)}\right)(\phi_t - 1)dN_t = Y_t(\phi_t - 1)dN_t$;
- v. $dM_t = dN_t - \lambda_t dt$.

According to the above five facts and general Ito's formula with jumps, ξ_t^ϕ is calculated as:

$$\begin{aligned} d\xi_t^\phi &= d(X_t Y_t) \\ &= dX_t Y_{t-} + X_{t-} dY_t + \Delta X_t \Delta Y_t \\ &= (1 - \phi_t)\lambda_t X_t Y_{t-} dt + X_{t-} Y_{t-} (\phi_t - 1) dN_t \\ &= (1 - \phi_t)\lambda_t \xi_{t-}^\phi + (\phi_t - 1)\xi_{t-}^\phi dN_t \\ &= (1 - \phi_t)\lambda_t \xi_{t-}^\phi + (\phi_t - 1)\xi_{t-}^\phi (dM_t + \lambda_t dt) \\ &= (\phi_t - 1)\xi_{t-}^\phi dM_t \end{aligned}$$

In the third equation, we made use of the fact that X_t is a continuous process which implies $\Delta X_t = 0$. Since M_t is a local martingale, we know that an integral against a local martingale is also a local martingale under certain regularity conditions (see Bremaud (1981)) for the integrand ■

Claim 2 : If ξ_t^ϕ is a martingale, then an equivalent martingale measure P^* is defined by $\frac{dP^*}{dP} = \xi_T^\phi$. Under this new martingale measure, N_t is still a nonexplosive counting process with intensity $\lambda_t\phi_t$.

Proof: To say that N_t is a counting process with intensity $\lambda_t\phi_t$ we need to show that $A_t = N_t - \int_0^t \lambda_s\phi_s$ is a P^* local martingale where $\frac{dP^*}{dP} = \xi_T^\phi$. Or equivalently, we can show that the process $Z_t = \xi_t^\phi A_t$ is a P local martingale.

By the first claim

$$dA_t = dN_t - \lambda_t\phi_t dt \quad \text{and} \quad d\xi_t^\phi = (\phi_t - 1)\xi_{t-} dM_t$$

Then by Ito's formula with jumps, we get

$$\begin{aligned} dZ_t &= d\xi_t^\phi A_{t-} + \xi_{t-}^\phi dA_t + \Delta\xi_t^\phi \Delta A_t \\ &= (\phi_t - 1)\xi_{t-}^\phi A_{t-} dM_t + \xi_{t-}^\phi (dN_t - \lambda_t\phi_t dt) + (\phi_t - 1)\xi_{t-} dN_t dN_t \\ &= (\phi_t - 1)\xi_{t-}^\phi A_{t-} dM_t - \xi_{t-}^\phi \lambda_t\phi_t dt + \phi_t \xi_{t-} dN_t \\ &= (\phi_t - 1)\xi_{t-}^\phi A_{t-} dM_t - \xi_{t-}^\phi \lambda_t\phi_t dt + \phi_t \xi_{t-} (dM_t + \lambda_t dt) \\ &= [(\phi_t - 1)\xi_{t-}^\phi A_{t-} + \phi_t \xi_{t-}] dM_t \end{aligned}$$

Hence, Z_t can be written as an integral against a local martingale, which would imply Z_t itself is a P local martingale. Therefore, A_t is a P^* local martingale and therefore N_t is a counting process with intensity $\lambda_t\phi_t$ under the new measure P^*

■

Appendix A

A.1 Extension, Index Process with Stochastic Volatility

Let us consider a more generalized version of the same problem where the intensity rate process depends upon the process of the underlying asset, and the underlying asset dynamics present stochastic volatility. The stochastic volatility process is defined as a positive bounded function of an OU process as proposed in Fouque et al. (2000b). Let us assume that $\lambda_t = g(X_t, S_t)$, where g represents a certain function, and X_t is some state process. Then, our SDE system for the prices will be

$$\begin{cases} dS_t &= \mu S_t dt + \sigma_t S_t dW_t^S \\ \sigma_t &= f(Y_t) \\ dY_t &= \alpha(m - Y_t)dt + \beta_\sigma dW_t^\sigma & dW_t^\sigma &= \rho_\sigma dW_t^S + \sqrt{1 - \rho_\sigma^2} dZ_t^\sigma \\ dX_t &= a(b - X_t)dt + \beta_\lambda dW_t^\lambda & dW_t^\lambda &= \rho_\lambda dW_t^S + \sqrt{1 - \rho_\lambda^2} dZ_t^\lambda \\ \lambda_t &= g(X_t, S_t) \end{cases} \quad (\text{A.1})$$

We would like to find a measure P^* under which the process $e^{-rt} S_t 1_{\{\tau > t\}}$ is a \mathcal{G}_t -martingale. As we showed earlier, this is equivalent to having the process $e^{-\int_0^t (r + \lambda_u^*) du} S_t$ as a \mathcal{G}_t -martingale, where λ_t^* is the intensity process under the measure P^* .

Using this and the two-step change of measure described in the notes, we obtain the system under consideration under P^* using the following change in the Brownian filtration

$$W_t^* = W_t + \int_0^t \theta_u du$$

where

$$W_t = [W_t^S \ Z_t^\sigma \ Z_t^\lambda] \quad \text{and} \quad \theta_t = \left[\frac{\mu - r - \phi_t g(X_t, S_t)}{f(Y_t)} \gamma_t \delta_t \right]$$

and the parameters γ , δ and ϕ are free.

Finally, the system becomes:

$$\begin{cases} dS_t &= (r + \phi_t g(X_t, S_t)) S_t dt + \sigma_t S_t dW_t^{*S} \\ \sigma_t &= f(Y_t) \\ dY_t &= [\alpha(m - Y_t) - \beta_\sigma (\rho_\sigma \frac{\mu - r - \lambda_t^*}{f(Y_t)} + \gamma_t \sqrt{1 - \rho_\sigma^2})] dt + \beta_\sigma dW_t^{*\sigma} \\ dX_t &= a(b - X_t) - \beta_\lambda (\rho_\lambda \frac{\mu_t - r - \phi_t g(X_t, S_t)}{f(Y_t)} + \delta_t \sqrt{1 - \rho_\lambda^2}) dt + \beta_\lambda dW_t^{*\lambda} \end{cases} \quad (\text{A.2})$$

Appendix B

B.1 Parameters Underlying the Stochastic Processes

In this subsection, we simply present the values of the parameters adopted to define the index process S , and the lambda processes, in the numerical examples involving correlation of default times (Figures 1 and 2), and first-to-default valuation (Figures 3 and 4). All figures were generated with 100,000 simulations.

Parameters	Figure 1	Figure 2	Figure 3	Figure 4
$r = \mu$	0.03	0.03	0.03	0.03
σ	0.1/0.25	0.1/0.25	0.25	0.25
S_0	100	100	100	100
a	1	1	1	1
b	0.03	0.03	0.1	0.1
β_X	0.01	0.01	0.1	0.1
X_0	0.03	0.03	0.1	0.1
N	2	2	5	5
ρ	0	0	varies	0
T	10	10	10	10
n	varies	varies	0	varies