Term Structure Movements Implicit in Asian Option Prices *

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Abstract

In this paper we implement dynamic term structure models that adopt bonds and Asian options in the estimation process. The goal is to analyze the pricing and hedging implications of term structure movements when options are (or not) included in the estimation process. We analyze how options affect the shape, risk premium and hedging structure of the dynamic factors. We find that the inclusion of options affects the loadings of the slope and curvature factors, and considerably changes the risk premium and hedging structure of all dynamic factors.

Keywords: Dynamic term structure models, latent factors, bond risk premium, Asian option pricing.
JEL classification: C51, G12.

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1 Introduction

Interest rate Asian options are securities depending on the accumulated value of the short-term rate. They are extremely useful hedging instruments for corporations with volatile periodic cash flows. Nevertheless, despite the existence of numerous theoretical results on the pricing of interest rate Asian options, previous research has been limited to cross-section option pricing\textsuperscript{1}. In this paper, in contrast, we estimate multi-factor dynamic term structure models with joint data on bonds and interest rate Asian options. Our contribution is to analyze how these options affect the shape, risk premium and hedging structure of the corresponding dynamic term structure factors.

We implement two versions of a three-factor Gaussian model, one including only bonds in the estimation process and the other including bonds and options. As a robustness check of the hypothesis of constant volatility, we also implement two versions of a three-factor affine model with one factor driving stochastic volatility ($A_1(3)$; Dai and Singleton, 2000). Closed-form formulas for bonds and Asian option prices allow efficient implementation of the Gaussian model. The $A_1(3)$ model is implemented via an adaptation of the Edgeworth expansion method proposed by Collin Dufresne and Goldstein (2002b) to price swaptions. The results of the $A_1(3)$ model qualitatively confirm the results obtained with the Gaussian model with respect to the shape of the term structure movements, and partially confirm the results concerning the risk premium structure.

The two versions of the Gaussian and $A_1(3)$ models are estimated respectively by maximum likelihood and quasi maximum likelihood methods. For each model, the first version adopts only bond data (the bond version), and the other combines bonds and data on at-the-money fixed-maturity options (the option version). Options appear to affect three dimensions of the dynamic model: the loadings of term structure movements, bond risk premium

decomposition and dynamic first-order hedging terms.

Empirical results show that the level is a robust factor common to both versions of each estimated model, while slope and curvature are less persistent under the option version of each model (see Figures 3 and 4). These movements have much higher mean reversion rates under the option version. Thus, while the information contained in bonds and at-the-money options agree on the main factor driving term structure movements, the information implicit in those option prices suggest faster variations for the secondary movements of the term structure.

Under the Gaussian model, the bond risk premium is slightly less volatile in the option version, and is more concentrated in the level factor. Under the $A_1(3)$ model, the level factor makes a small contribution to the term structure of risk premiums in both versions, while the slope factor appears as the main factor driving the premium. Although each model has its own risk premium decomposition, both agree that when switching from the bond version to the option version of each model, level has a greater effect on the risk premium decomposition while curvature has a smaller effect.

A comparison of the two estimated versions of the Gaussian model further reveals that the bond version better captures the term structure of bond yields, as expected. However the bond version is outperformed by the option version in the option pricing and hedging exercises. From a hedging perspective, the bond version is only able to capture 5.10% of the price movements of the at-the-money option adopted, in contrast with 94.74% for the option version. Analysis of the dynamic hedging weights attributed to each factor under each version shows clearly that both versions give no importance to the curvature dynamic factor when hedging the at-the-money option. On the other hand, level and slope weights are much more volatile under the option version of the model.

Related works include the papers by Umantsev (2001), Bikbov and Cher-

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2To reduce the size and improve the organization of the paper, for the $A_1(3)$ model we only report results related to shape and risk premium structure of dynamic factors, but not those related to dynamic hedging. We should clarify, however, that once the model is implemented, the hedging analysis can be naturally pursued without incurring higher computational costs.

3Note that this was expected since the option version is perfectly pricing this option, and the 4.79% variability of prices not captured in the delta-hedge is due to second-order effects. The only reason to provide hedging results under the option version is to allow comparison of dynamic hedging weights across versions.
nov (2004), Li and Zhao (2006), Joslin (2007) and Almeida and Vicente (2009). Bikbov and Chernov (2004) use a joint dataset of Eurodollar bonds and options to economically discriminate among affine models with different volatility specifications. While they test how including options affects the shape of term structure factors, they do not present a risk premium analysis or any dynamic hedging analysis. Umantsev (2001) estimates three-factor affine dynamic term structure models simultaneously adopting swaps and swaption prices to analyze the risk premium structure. However, like Bikbov and Chernov (2004), he does not provide an analysis of the hedging performance of the models. In contrast, Li and Zhao (2006) implement quadratic term structure models (Ahn et al., 2002) to test their hedging performance with respect to cap derivatives. However their dynamic models are estimated based on only bond data, while caps are considered as separate instruments to test hedging performance. In contrast, we explicitly include options in our estimation process. Joslin (2007) implements four-factor affine models with a flexible covariance structure that allows simultaneously pricing bonds and swaptions. He analyzes the hedging implications of such models finding that dynamic hedging strategies using bonds alone produce reasonably good hedges for derivative positions. While he focuses on the dynamic hedging properties of his models based on swaption data, we look at different aspects (including hedging) of how interest rate Asian options affect term structure movements.

The main innovation of our work is to provide an empirical analysis of term structure movements based on dynamic models estimated with Asian options data, which appear to be of interest on their own account. In this sense, there is only one work which is close in spirit to the present paper: Almeida and Vicente (2009), who implement a dynamic term structure model with bonds and Asian options to analyze the volatility risk premium structure of these joint markets. The dynamic term structure model implemented belongs to the class of unspanned stochastic volatility models (USV; Collin Dufresne and Goldstein, 2002a), generating an incomplete bond market. For this specific reason, the only way to estimate their proposed USV model is with the use of joint bonds and options data. On the other hand, in the present paper we analyze dynamic term structure models generating complete bond markets. Thus we are able to estimate different versions of each dynamic model, some including only bond data, while others include both bond and option data. This ability to estimate different versions is fundamental here since we are interested in contrasting the different versions of each
dynamic model with respect to how they affect term structure movements.

In summary, from a theoretical viewpoint we provide efficient ways of implementing multi-factor affine term structure models (Gaussian and $A_1(3)$) including interest rate Asian options in the estimation. From an empirical standpoint, our contribution is to provide an examination of how including Asian options in the estimation process of those dynamic models will affect the loadings, the risk premium structure and first-order hedging of term structure movements. Our results should be useful for risk management and portfolio management purposes, and as a tool for practitioners to quickly price bonds and Asian options with analytical formulas.

The paper is organized as follows. Section 2 describes the market of ID-futures (bonds), and IDI interest rate Asian options. Section 3 presents the model, the pricing of zero-coupon bonds and IDI options, and the first-order dynamic hedging properties of these options. Section 4 describes and implements the estimation process in each version. Section 5 compares the two dynamic versions of the Gaussian model, considering the empirical dimensions described above. Results on the $A_1(3)$ model regarding the shape and risk premium structure of term structure movements are also presented. Section 6 concludes. Appendix A contains theoretical results on the pricing of fixed income instruments under the Gaussian model. Appendix B show how we price bonds and interest rate Asian options under the $A_1(3)$ model. Finally, Appendix C presents a detailed description of the maximum likelihood estimation procedure used here.

2 Data and Market Description

2.1 ID-futures

The one-day inter bank deposit future contract (ID-future) with maturity $T$ is a future contract whose underlying asset is the accumulated daily ID rate\textsuperscript{4} capitalized between the trading time $t$ ($t \leq T$) and $T$. The contract size corresponds to R$ 100,000.00 (one hundred thousand Brazilian Reais) discounted by the accumulated rate negotiated between the buyer and the seller of the contract.

\textsuperscript{4}The ID rate is the average one-day inter-bank rate, calculated by the clearinghouse CETIP (Center for Custody and Financial Settlement of Securities) every business day. The ID rate is expressed as the effective rate per annum, based on 252 business days.
This contract is very similar to a zero-coupon bond, except that it pays margin adjustments every day. Each daily cash flow is the difference between the settlement price\(^5\) on the current day and the settlement price on the day before, corrected by the ID rate of the previous day.

The Brazilian Mercantile and Futures Exchange (BM&F) is the entity that offers the ID-future. The number of authorized contract-maturity months is fixed by the BM&F (on average, there are about twenty authorized contract-maturity months for each day, but only around ten are liquid). Contract-maturity months are the first four months subsequent to the month in which a trade was made, and after that the first months of each following quarter. The expiration date is the first business day of the contract-maturity month.

2.2 ID Index and its Option Market

The ID index (IDI) is defined as the accumulated ID rate. If we associate the continuously-compounded ID rate to the short term rate \( r_t \) then

\[
IDI_t = IDI_0 \cdot e^{\int_0^t r_u \, du}.
\]  

This index, computed on every business day by the BM&F, has been adjusted to a value of 100,000 points in January 2, 1997, and has actually been reset to its initial value in January 2, 2003.

An IDI option with maturity \( T \) is a European option where the underlying asset is the IDI and whose payoff depends on \( IDI_T \). When the strike is \( K \), the payoff of an IDI option is \( L_c(T) = (IDI_T - K)^+ \) for a call and \( L_p(T) = (K - IDI_T)^+ \) for a put. For more details about IDI options, see Brace (2008).

As can be noticed, IDI options have a peculiar characteristic which is not shared by usual international options: they are Asian options. Their payoff depends on the integral of the short-term rate through the path between the trading date \( t \) and the option maturity date \( T \). This makes them particularly suited to complement the theoretical papers on interest rate Asian options that are discussed in Section 1. Moreover, Asian options are popular over-the-counter instruments that are cheaper than their vanilla counterparts (caps, floors), less subjective to price manipulation, and offer simpler hedging

\(^5\) The settlement price at time \( t \) of an ID-future with maturity \( T \) is equal to R$ 100,000.00 discounted by its closing price quotation.
strategies than regular interest rate options (see Longstaff, 1995 and Chacko and Das, 2002).

The BM&F is also the trading venue for IDI call options. Strike prices (expressed in index points) and the number of authorized contract-maturity months are established by the BM&F. Contract-maturity months can be any month, and the expiration date is the first business day of the maturity month. Usually there are 30 authorized series within each day, of which about a third are liquid.

2.3 Data

Our data consist of time series of ID-future yields for all different liquid maturities, and prices of IDI options for different strikes and maturities, covering the period from January 2003 to December 2005.

The BM&F maintains a daily historical database with prices and number of trades for all ID-futures and IDI options that have been traded within a day. Yields of zero-coupon bonds with fixed maturities are estimated with a cubic interpolation scheme applied to the ID-futures dataset. In estimating the models, we use the yields from bonds with fixed maturities of 1, 21, 63, 126, 189, 252 and 378 business days. Figure 1 presents the evolution of some bond yields extracted from ID-futures data, from January 2003 to December 2005. Yields range from a maximum of 25% at the beginning of the sample period to a minimum of 15% in February 2004.

Regarding options, we use two different databases. The first is composed of an at-the-money fixed-maturity IDI call, with time to maturity of 95 business days. The second is composed by choosing within each day the most liquid IDI call. We use the first database to estimate the dynamic models (option versions), and the second to test the pricing performance of the two versions. As hedging cannot be tested with the database on the most liquid IDI options because moneyness and maturity change through time, we carried out the hedging using the at-the-money options of the first

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6There are transactions in this market with longer maturities (up to ten years) but the liquidity is considerably lower. The maturities of 21, 63, 126, 189, 252 and 378 business days correspond, respectively, to 1, 3, 6, 9, 12 and 18 months.

7Moneyness is defined as the ratio of the present strike value over the current IDI value.

8The at-the-money IDI call prices are obtained by an interpolation of Black implied volatilities in a similar procedure to that adopted to construct the original VIX volatilities.
Table 1 presents descriptive statistics of these two databases. Note that the most liquid options are in-the-money and have average time to maturity greater than 95 business days. Therefore, their average price is greater than the average price of the at-the-money IDI calls.

After excluding weekends, holidays, and no-trading business days, there are 748 daily observations of yields from zero-coupon bonds and option prices.

3 The Model

The uncertainty in the economy is characterized by a filtered probability space \((\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, \mathbb{P})\). The existence of a pricing measure \(Q\) under which discounted asset prices are martingales is assumed. Following Duffie and Kan (1996) and Dai and Singleton (2000), we adopt multiple factors to drive the uncertainty of the yield curve. The model is specified through the definition of the short-term rate \(r_t\) as a sum of \(N\) Gaussian random variables:

\[
r_t = \phi_0 + \sum_{i=1}^{N} X_t^i,
\]

where the dynamics of process \(X\) is given by

\[
dX_t = -\kappa X_t dt + \rho dW_t^Q,
\]

with \(W^Q\) being an \(N\)-dimensional Brownian motion under \(Q\), \(\kappa\) is a diagonal matrix with \(\kappa_i\) in the \(i_{th}\) diagonal position, and \(\rho\) is a matrix responsible for correlation among the \(X\) factors. The connection between the martingale probability measure \(Q\) and objective probability measure \(\mathbb{P}\) is given by Girsanov’s Theorem, with an essentially affine (Duffee, 2002) market price of

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9In this case it is clear that the option version will outperform the bond version, since the first perfectly prices the at-the-money option. However, as explained in the empirical section, the most interesting aspect of this hedging exercise is to compare the dynamic allocations provided to each term structure movement by each model.

10This sample size is compatible with that found in other recent studies containing derivatives data from emerging economies (see for instance, Pan and Singleton, 2008). In addition, as our study contains high frequency data, the number of observations (748) adopted to estimate the dynamic term structure model is large enough to avoid small-sample biases.

11Constrained for admissibility purposes (see Dai and Singleton, 2000).
risk
\[ dW^P_t = dW^Q_t - \lambda_X X_t dt, \tag{4} \]
where \( \lambda_X \) is an \( N \times N \) matrix and \( W^P \) is a Brownian motion under \( \mathbb{P} \).

In the next three subsections we analyze the pricing of bonds and options and the hedging strategy under the Gaussian model. To check the robustness of the homocedastic hypothesis, we also implement a model (A1(3)) with one dynamic factor driving the volatilities of yields. Since the stochastic volatility model is not the core of this work, the details of the A1(3) model are presented in Appendix B.

### 3.1 Pricing Zero-Coupon Bonds

Let \( P(t, T) \) denote the time \( t \) price of a zero-coupon bond maturing at time \( T \), paying one monetary unit. It is known that multi-factor Gaussian models offer closed-form formulas for zero-coupon bond prices. The next two lemmas present a simple proof of this fact for the particular model at hand.

**Lemma 1** Let \( y(t, T) = \int_t^T r_u du \). Then, under measure \( Q \) and conditional on the sigma field \( \mathcal{F}_t \), \( y \) is normally distributed with mean \( M(t, T) \) and variance \( V(t, T) \) given by
\[
M(t, T) = \phi_0 \tau + \sum_{i=1}^N \frac{1 - e^{-\kappa_i \tau}}{\kappa_i} X^i_t
\tag{5}
\]
and
\[
V(t, T) = \sum_{i=1}^N \frac{1}{\kappa_i^2} \left( \tau + \frac{2}{\kappa_i} e^{-\kappa_i \tau} - \frac{1}{2\kappa_i} e^{-2\kappa_i \tau} - \frac{3}{2\kappa_i} \right) \sum_{j=1}^N \rho^2_{ij} + \\
+ 2 \sum_{i=1}^N \sum_{k>i} \frac{1}{\kappa_i \kappa_k} \left( \tau + \frac{e^{-\kappa_i \tau} - 1}{\kappa_i} + \frac{e^{-\kappa_k \tau} - 1}{\kappa_k} - \frac{e^{-(\kappa_i + \kappa_k) \tau} - 1}{\kappa_i + \kappa_k} \right) \sum_{j=1}^N \rho_{ij} \rho_{kj},
\tag{6}
\]
where \( \tau = T - t \).

**Proof.** See Appendix A. \( \blacksquare \)

**Lemma 2** The price at time \( t \) of a zero-coupon bond maturing at time \( T \) is
\[
P(t, T) = e^{A(\tau) + B(\tau)' X_t},
\tag{7}
\]
where \( A(\tau) = -\phi_0 \tau + \frac{1}{2} V(t, T) \) and \( B(\tau) \) is a column vector with \(-\frac{1-e^{-\kappa_i \tau}}{\kappa_i}\) as the \( i \)th element.
Proof. See Appendix A. ■

Using (7) and Itô’s lemma, we can obtain the dynamics of bond prices under the martingale measure \( Q \)

\[
\frac{dP(t, T)}{P(t, T)} = r_t dt + B(\tau)' \rho dW^Q_t. \tag{8}
\]

To hold this bond, investors will ask for an instantaneous expected excess return. Then, under the objective measure, the bond price dynamics is

\[
\frac{dP(t, T)}{P(t, T)} = (r_t + z^i(t, T)) dt + B(\tau)' \rho dW^P_t. \tag{9}
\]

Applying Girsanov’s Theorem to change measures, the instantaneous premium is obtained as

\[
z^i(t, T) = B(\tau)' \rho \lambda X_t. \tag{10}
\]

3.2 Pricing Interest Rate Asian Options

IDI options are continuous-time interest rate Asian options. Theoretical results and cross-section pricing of interest rate Asian options can be found in Geman and Yor (1993), Longstaff (1995), Leblanc and Scaillet (1998), Cheuk and Vorst (1999), Bakshi and Madan (1999), Chacko and Das (2002), and Dassios and Nagaradjasarma (2003). Each of these papers builds on different techniques, including Fourier transforms, representations in series of functions and Bessel processes theory. In this section, we propose analytical formulas for Asian option prices that allow for efficient implementation of the dynamic term structure model, thus empirically complementing the above-mentioned theoretical works.

Denote by \( c(t, T) \) the time \( t \) price of a call option on the IDI, with maturity \( T \) and strike price \( K \). Then

\[
c(t, T) = \mathbb{E}^Q \left[ e^{-\int_t^T r_u du} \max(\text{IDI}_T - K, 0) | \mathcal{F}_t \right] =
\]

\[
= \mathbb{E}^Q \left[ \max(\text{IDI}_t - Ke^{-y(t, T)}, 0) | \mathcal{F}_t \right]. \tag{11}
\]

**Lemma 3** The price at time \( t \) of the above-mentioned option under the Gaussian model is

\[
c(t, T) = \text{IDI}_t \Phi(d) - K P(t, T) \Phi(d - \sqrt{V(t, T)}), \tag{12}
\]

where $\Phi$ denotes the cumulative normal distribution function, and $d$ is given by

$$d = \frac{\log \frac{IDI}{K} - \log P(t, T) + V(t, T)/2}{\sqrt{V(t, T)}}.$$  \hspace{1cm} (13)

Proof. See Appendix A. \hfill \blacksquare

If $p(t, T)$ is the price at time $t$ of the IDI put with strike $K$ and maturity $T$, then by the put-call parity

$$p(t, T) = KP(t, T)\Phi(\sqrt{V(t, T)} - d) - IDI_t \Phi(-d).$$  \hspace{1cm} (14)

### 3.3 Hedging IDI Options

When hedging an instrument, we are interested in the composition of a portfolio which approximately neutralizes variations in the price of this instrument. To that end, we should make use of a set of additional instruments that present dynamics related to the targeted instrument. Alternatively, it is known that each state variable driving uncertainty in the term structure is responsible for one type of movement. These movements are represented by the state variable loadings as a function of time to maturity (see Section 5 for a concrete example). As in Li and Zhao (2006), here we assume that these state variables are tradable assets which can be used as instruments to compose the hedging portfolio. The main advantage of this approach is to avoid introduction of additional sources of error due to approximate relations between the hedging instruments and the state variables.

The goal of this hedging analysis is to identify whether the bond version of the model captures the dynamics of IDI options. A delta hedging procedure is performed by equating the first derivatives (with respect to state variables) of the hedging portfolio to the first derivatives (with respect to state variables) of the instrument being hedged. We chose this instrument, for illustration purposes, to be one contract of a call on the IDI index with strike $K$ and maturity $T$. Letting $\Pi_t$ denote the time $t$ value of the hedging portfolio, by assumption it must satisfy

$$\Pi_t = q_1^t X_1^t + q_2^t X_2^t + \ldots + q_N^t X_N^t,$$  \hspace{1cm} (15)

where $q_i^t$ is the number of units of $X_i^t$ in the hedging portfolio, and $X_i^t$ is the $i^{th}$ term structure dynamic factor. By simply equating the first-order
variation of $\Pi_t$ to the first-order variation of the IDI option price $c(t, T)$, the result is $q^i_t = \frac{\partial c(t, T)}{\partial X^i_t}$. From calculating the partial derivatives using (12) it follows that
\[
q^i_t = \frac{1 - e^{-\kappa_i \tau}}{\kappa_i \sqrt{V(t, T)}} \left[ IDI_t \Phi'(d) + KP(t, T) \sqrt{V(t, T)} \Phi(d - \sqrt{V(t, T)}) - KP(t, T) \Phi'(d - \sqrt{V(t, T)}) \right].
\]
(16)

In the empirical exercise presented below, (16) is used to readjust the hedging on a daily basis.

4 Parameter Estimation

In this section we estimate two versions of a three factor Gaussian model\(^{12}\). We obtain the model parameters based on the maximum likelihood procedure adopted by Chen and Scott (1993). Appendix C gives more details about this procedure and describes the quasi maximum likelihood method used to estimate the $A_1(3)$ model.

In the bond version, only ID-futures data, in the form of fixed maturity zero-coupon bond implied yields, are used in the estimation process. Bonds with maturities of 1, 126, and 252 business days are observed without error\(^{13}\). For each fixed $t$, the state vector is obtained by solving the following linear system:
\[
rb_t(0.00397) = -\frac{A(0.00397, \phi)}{0.00397} - \frac{B(0.00397, \phi)'}{0.00397} X_t
\]
\[
rb_t(0.5) = -\frac{A(0.5, \phi)}{0.5} - \frac{B(0.5, \phi)'}{0.5} X_t
\]
\[
rb_t(1) = -\frac{A(1, \phi)}{1} - \frac{B(1, \phi)'}{1} X_t.
\]
(17)

where $rb_t$ represents the vector of ID yields observed at time $t$ and $\phi$ is a vector stacking the model parameters.

\(^{12}\)According to a principal component analysis applied to the covariance matrix of observed yields, three factors are sufficient to describe 99.5% of the variability of the term structure of ID bonds.

\(^{13}\)We also tested inversions of the state vector considering other combinations of bonds, obtaining similar qualitative results with regard to parameter estimation and bond pricing errors.
Bonds with maturities of 21, 63, 189 and 378 business days are assumed to be observed with Gaussian errors $u_t$ uncorrelated in the time dimension:

$$rb_t(0.0833) = -\frac{A(0.0833, \phi)}{0.0833} - \frac{B(0.0833, \phi)'}{0.0833}X_t + u_t(0.0833)$$

$$rb_t(0.25) = -\frac{A(0.25, \phi)}{0.25} - \frac{B(0.25, \phi)'}{0.25}X_t + u_t(0.25)$$

$$rb_t(0.75) = -\frac{A(0.75, \phi)}{0.75} - \frac{B(0.75, \phi)'}{0.75}X_t + u_t(0.75)$$

$$rb_t(1.5) = -\frac{A(1.5, \phi)}{1.5} - \frac{B(1.5, \phi)'}{1.5}X_t + u_t(1.5).$$

The Jacobian matrix is

$$Jac_t = \begin{bmatrix}
-\frac{B(0.00397, \phi)'}{0.00397} \\
-\frac{B(0.5, \phi)'}{0.5} \\
-\frac{B(1, \phi)'}{1}
\end{bmatrix}.$$  \hfill (19)

In the option version, options are included in the estimation procedure. This is done by assuming that the instruments observed without error are bonds with maturities of 1 and 189 business days and an at-the-money IDI call option with maturity of 95 business days, whose time $t$ observed price is denoted by $cs_t$. The state vector is obtained by solving the following nonlinear system

$$rb_t(0.00397) = -\frac{A(0.00397, \phi)}{0.00397} - \frac{B(0.00397, \phi)'}{0.00397}X_t$$

$$rb_t(0.75) = -\frac{A(0.75, \phi)}{0.75} - \frac{B(0.75, \phi)'}{0.75}X_t$$

$$cs_t = c(t, t + 0.377),$$

where $c(t, T)$ is given by Equation (11).

Bonds with maturities of 21, 63, 252, and 378 business days are priced...
with uncorrelated Gaussian errors $u_t$:

$$rb_t(0.0833) = -\frac{A(0.0833,\phi)}{0.0833} - \frac{B(0.0833,\phi)'X_t}{0.0833} + u_t(0.0833)$$

$$rb_t(0.25) = -\frac{A(0.25,\phi)}{0.25} - \frac{B(0.25,\phi)'X_t}{0.25} + u_t(0.25)$$

$$rb_t(1) = -\frac{A(1,\phi)}{1} - \frac{B(1,\phi)'X_t}{1} + u_t(1)$$

$$rb_t(1.5) = -\frac{A(1.5,\phi)}{1.5} - \frac{B(1.5,\phi)'X_t}{1.5} + u_t(1.5).$$

The Jacobian matrix is

$$Jac_t = \begin{bmatrix} -\frac{B(0.00397,\phi)'}{0.00397} \\ -\frac{B(0.75,\phi)'}{0.75} \\ q_t \end{bmatrix},$$

where $q_t = [q_t^1, \ldots, q_t^N]$ with $q_t^i$ calculated for $T = t + 0.377$ (see (16)).

In both versions of the model, the transition probability $p(X_t|X_{t-1}; \phi)$ is a three-dimensional Gaussian distribution with known mean and variance as functions of parameters appearing in $\phi$.

Tables 2 and 3 present, respectively, the values of the parameters estimated for each version of the model. Standard deviations are obtained by the BHHH method (see Davidson and MacKinnon, 1993). In both versions, most of the parameters are significant at a 95% confidence interval, except for a few risk premium parameters, and one parameter which comes from the correlation matrix of the Brownian motions. The long-term short-rate mean $\phi_0$ was fixed equal to 0.18, compatible with the ID short-rate sample mean of 0.1778\(^{14}\).

### 5 Empirical Results

Table 4 presents the mean absolute errors of the yields and principal components (level, slope and curvature) in both versions of the Gaussian and $A_1(3)$

\[^{14}\text{We also tested optimization including this parameter, but the results had higher standard errors for a considerable fraction of the parameter vector.}\]
models. For the bond version of the Gaussian model, the mean absolute error of zero-coupon bond yields with maturities of 21, 63, 189 and 378 business days are, respectively, 18.10 bps, 6.93 bps, 1.76 bps and 11.52 bps. Standard deviations of these errors, which provide a metric for their time series variability, are 24.52 bps, 9.52 bps, 2.26 bps and 14.07 bps. For the option version, the mean absolute error of bonds yields with maturities of 21, 63, 252 and 378 business days are, respectively, 29.72 bps, 14.89 bps, 12.93 bps and 39.03 bps, with standard deviations of 35.37 bps, 17.70 bps, 15.92 bps and 46.54 bps. The absolute errors of the principal components are also greater in the option version, as expected. Figure 2 shows the average observed and model implied term structures of interest rates for zero-coupon bonds in each estimated version of the Gaussian model. It is clear from this figure that on the pricing of bonds, the bond version outperforms the option version.

The $A_1(3)$ model presents very similar pricing errors as the Gaussian model, in both versions. For the bond version, the mean absolute error of zero-coupon bond yields with maturities of 21, 63, 189 and 378 business days are respectively 18.34 bps, 7.21 bps, 1.86 bps and 11.75 bps, with values for all maturities approximately 0.2 bps above the Gaussian model. The standard deviations of these errors are respectively 24.58 bps, 9.62 bps, 2.29 bps and 14.07 bps, practically equal to their Gaussian counterpart values. For the option version of the $A_1(3)$ model, the mean absolute error of yields with maturities of 21, 63, 252 and 378 business days are, respectively, 25.84 bps, 25.21 bps, 11.90 bps and 36.89 bps, with standard deviations of 36.02 bps, 36.20 bps, 15.95 bps, and 46.38 bps. Note that the Gaussian and $A_1(3)$ models differ only in the error of the 63-business day yield in the option version.

In order to estimate the errors of the principal components, we first obtain the time series of the principal components using the full database of yields. Next we evaluate the implied components in each version of the models using the implied yields and the same rotation obtained in the first step.

Bps stands for basis points. One basis point is equivalent to 0.01%.

Under the bond version, the three dimensional latent vector $X$, characterizing uncertainty in the economy, is fully inverted from bond data. In contrast, the option version only captures the yields of two bonds without errors, because the third instrument priced without error is an at-the-money option.

Probably because the option 95-business day maturity might be affecting the CIR process, asymmetrically driving volatility in the $A_1(3)$ model (when compared to the Gaussian model), slightly distorting the pricing of bonds with maturities close to the 378 day maturity.
Whatever the model used, the option versions consistently price yields and principal components worse. Although in the option versions we use fewer bond yields in the estimation procedure, the larger errors suggest that a four-factor affine model would be more appropriate, as pointed out by Joslin (2007).

5.1 Term Structure Movements and Bond Risk Premiums

Figure 3 presents the loadings of the three dynamic factors under each version of the Gaussian model (solid lines correspond to the bond version, dotted lines to the option version). The level factor\textsuperscript{19} presents loadings indistinguishable across versions. However, slope and curvature factors are clearly different. They both have higher curvatures in the option version, suggesting that option investors tend to react faster (than bond investors) to news that affect the term structure of bond risk premiums in an asymmetric way\textsuperscript{20}. Similarly, Figure 4 presents the loadings of the three dynamic factors in each version of the $A_1(3)$ model. Note the similarity between Figures 3 and 4, indicating that the modification in the shape of term structure dynamic factors when Asian options are included in the estimation process affects the Gaussian model and the $A_1(3)$ model in closely related ways. In the $A_1(3)$ model the mean reversion rate of the curvature factor is slightly less affected when options are included. In fact, while in the Gaussian model the mean reversion rate of the curvature factor increases from 6.34 (bond version) to 37.63 (option version), in the $A_1(3)$ model it increases from 6.84 to 15.87. Figure 5 presents the state variables driving each term structure movement, for the two versions of the Gaussian model\textsuperscript{21}. Note that the time series of the slope and curvature factors, in the option version, have spikes that are consistent with fast mean reverting variables\textsuperscript{22}.

\begin{footnotesize}
\begin{itemize}
\item $^19$It is the one with slowest mean reversion speeds and responsible for explaining most of the variation in yields.
\item $^20$Note that a shock in the level factor affects the risk premium term structure symmetrically.
\item $^21$The average value of the short-rate ($\phi_0$) should be added to the level state variable in order to obtain the level factor.
\item $^22$Results for the time series of dynamic factor are very similar under the $A_1(3)$ model, and are available upon request.
\end{itemize}
\end{footnotesize}
An important point related to the modification of term structure movements is to understand the implications for investors’ interpretation of risks when options are or not included in the estimation process. This can be addressed in at least two ways: by observing the time series of the model’s implied bond risk premiums and contrasting across versions, or by directly observing bond risk premium decomposition as a combination of term structure movements, in each version.

Figure 6 presents graphs of the term structures of the instantaneous bond risk premium (measured by (10)) at different instants, for the Gaussian model. Note that the cross section of premiums is very distinct across versions, and in particular the longer the maturity the larger is the difference between the risk premium implied by each version. In addition, in the option version, the term structure of risk premiums is better approximated by a linear function, and the risk premiums are in general lower. The time series behavior of the premiums can be better observed in Figure 7, which presents the evolution of the instantaneous risk premium for the 1-year bond, in the two versions. During the period from September 2003 to December 2004, the premium is significantly higher under the bond version. That was a period when interest rates were consistently being lowered by the Central Bank of Brazil. In this context the smaller premium (under the option version) indicates the possibility of inertia by bond investors in re-estimating their expectations of the long-term behavior of interest rates, as opposed to a faster reaction of option market players.

The risk premium decomposition across movements of the term structure provides a direct way of identifying the shifts in importance of factors when options are included in the estimation process. From (10), it is clear that the risk premium is a linear combination of the state variables: $z(t, t+\tau) = a_1(\tau)X_{t}^1 + a_2(\tau)X_{t}^2 + a_3(\tau)X_{t}^3$. Figure 8 presents, for the Gaussian model, the term structure of risk premiums decomposed for each maturity among the three movements: level, slope and curvature. Solid lines represent the bond version and dashed lines the option version. For each fixed maturity, the sum of the absolute weights on the three movements is 100%. The decomposition presents a clearly distinct pattern for maturities shorter and longer than 0.5 year, in both versions. For instance, in the bond version, the curvature factor explains more than 70% of the premium for short maturities while curvature and slope together explain the premium for longer maturities. In the option version the level factor explains most of the premium for longer maturities while it shares this role with the curvature factor.
for shorter maturities. In both versions the slope contributes negatively to the risk premium decomposition. In general, the risk premium is more sensitive to the curvature and slope factors in the bond version, and to the level and curvature factors in the option version. By contrasting factor loadings and risk premiums, it is possible to identify that the use of options data provides less persistent slope and curvature movements, but prices the most persistent factor (level). On the other hand, when only bonds are used in the estimation process, secondary movements (slope and curvature) are more persistent, but are priced instead of the level movement (still the most persistent factor). The results suggest that in the Brazilian fixed income market, options investors are more concerned with monetary policy through interest rate levels, while bond investors are more concerned with the volatility of interest rates through curvature and slope (see Litterman et al., 1991).

The corresponding risk premium decomposition in the $A_1(3)$ model is presented in Figure 9. Similarly to the Gaussian model, when options are included in the estimation the importance of the level factor in the risk premium decomposition increases, while the importance of the curvature factor decreases. However, the two models disagree on how to price the slope factor. While under the Gaussian model the slope has less influence on the risk premium when options are included, in the $A_1(3)$ model the opposite appears to happen. This might be a consequence of stochastic volatility in the $A_1(3)$ model, driven by the level factor, generating tension between first and second conditional moments. This has been previously observed in the literature on affine dynamic models (see Duffee, 2002 and Duarte, 2004). In any case, we are not advocating that the changes when Asian options are included in the estimation process should be robust to changes in the dynamic term structure model chosen. The interesting point is that both Gaussian and $A_1(3)$ models appear to agree on enough points to allow a fixed income manager to safely consider implementing the simpler Gaussian model (instead of $A_1(3)$) to extract information on shapes and risk premium structure of term structure movements in joint bond/interest rate Asian option markets.

### 5.2 Pricing and Hedging Options

The goal of the next exercise is to understand how useful the inclusion of options can be in the estimation process of the dynamic model when pricing and hedging options. Since in the option version an at-the-money option
is used to invert the state vector, this exercise is only interesting if out-of-sample options are adopted. For this reason, we use the database of the most liquid IDI call options when comparing pricing performances across versions.

Figure 10 presents the observed option prices versus those estimated by the model. The points represent the bond version and x’s the option version, in the Gaussian model. For modeling purposes, an ideal relation would be a 45 degree line passing through the origin (solid line in Figure 10). In the bond version, a linear regression of observed prices depending on model prices, produces an $R^2 = 97.5\%$, an angular coefficient of 1.0423 (p-value < 0.01) and a linear coefficient of 86.83 (p-value < 0.01). The high $R^2$ indicates that the option prices obtained in the bond version correctly capture the time series variability of observed option prices (high correlation). However, the high value of the linear coefficient implies that the bond version consistently underestimates option prices. The underestimation of option prices is confirmed by Figure 11, which shows the relative error defined by model price minus observed price, divided by observed price. Note how in the bond version it is smaller than zero most of the time. The absolute relative pricing error has an average of 17.53%.

When the same regression is performed for the option version, the $R^2$ is slightly lower, at 97.2\%, probably due to some mispricing of options with prices in the range [1500,3000] (see Figure 10). On the other hand, both the angular coefficient of 1.0121 (p-value < 0.01) and the linear coefficient of 11.67 (p-value = 0.14) are closer to ideal values. The smaller linear coefficient indicates that when options are included in the estimation process, they help the dynamic model to better capture the level of option prices. The dotted line in Figure 11 shows the relative pricing error for the option version. Note that it clearly outperforms the bond version, except for the end of the sample period when it overestimates option prices. It achieves an average absolute value of 10.75\%, a 40\% improvement over the bond version.

The next step implements a dynamic delta-hedging strategy on the fixed-maturity at-the-money IDI call option\textsuperscript{24}. Note that if the hedging is effective, variations in the hedging portfolio should approximately offset variations in the option price. The correlation coefficients between these variations are

\textsuperscript{23}For comparison purposes, see Jagannathan et al. (2003), who price U.S. caps applying a three-factor CIR model estimated with U.S. Libor and swap data.

\textsuperscript{24}In the hedging analysis we adopted a fixed-maturity, fixed-moneyness option. Otherwise changes in prices would reflect not only the price dynamics but also changes in the type of the option.
5.10% and 94.74% for the bond and option versions respectively, suggesting that the option-based version is much more efficient when hedging. In fact, one could expect with no surprises that the option version would be able to hedge well since the at-the-money option is inverted to extract the state vector. In this sense, the hedging error for the option model is essentially a second-order error not captured by the delta-hedging procedure. However, the result of interest is the comparison of dynamic hedging weights across versions. Figure 12 depicts the number of units in the hedging portfolio invested in each state variable. Observe that in both versions of the model the option is more sensitive to the level factor and less sensitive to the curvature factor, and in particular under the option version, the allocations to both level and slope factors are much more volatile. This high volatility of the allocations reflects the fact that at-the-money options are highly sensitive to changes in their underlying assets, which in this case are interest rates.

6 Conclusion

Asian options are important over-the-counter instruments that are very valuable hedging resources. To understand how these options affect corresponding bond markets, we used the technology of affine processes to implement different dynamic models including Asian options in the estimation process. In particular, with the use of analytical formulas for bonds and Asian options, two versions of a dynamic multi-factor Gaussian model were estimated, one including only bond data (bond version) and the other combining bonds and interest rate Asian option data (option version). The main interest was to verify if and how Asian options change the loadings, risk premium and hedging structures of dynamic term structure factors.

We found that interest rate Asian options bring information that primarily affects the speed of mean reversion of the slope and curvature of the yield curve, and that strongly affects the decomposition of bond risk premiums. In addition, when delta-hedging an at-the-money option, both implemented versions of the Gaussian model gave small importance to the curvature factor, while the option version presented much more volatile weights on slope and level factors. These seem to be necessary to capture the dynamics of option prices.

We also implemented a model generating stochastic volatility for interest rates to verify the validity of the results obtained with the Gaussian model.
The stochastic volatility model obtained very similar results (to the Gaussian model) regarding the effects of Asian options on the loadings of term structure movements, and partially confirmed the results about the way options change the bond risk premium structure. Our results complement theoretical studies on the pricing of interest rate Asian options, because we pioneered implementations of dynamic models that include these options in the estimation process. They should be useful for portfolio and risk management purposes as simple and effective tools for pricing and hedging fixed income instruments with the use of interest rate Asian options.
Appendix A

Proof. Lemma 1
By Itô’s rule, for each $t < T$ the unique strong solution of (3) is

$$X_t^i = X_t^i e^{-\kappa_i (T-t)} + \sum_{j=1}^N \rho_{ij} \int_t^T e^{-\kappa_i (T-s)} dW_s^j, \quad i = 1, \ldots, N.$$  

Then

$$r_T = \phi_0 + \sum_{i=1}^N \left( X_t^i e^{-\kappa_i (T-t)} + \sum_{j=1}^N \rho_{ij} \int_t^T e^{-\kappa_i (T-s)} dW_s^j \right).$$

Stochastic integration by parts implies that

$$\int_t^T X_u^i \, du = \int_t^T (T-u) \, dX_u^i + (T-t) X_t^i. \quad (22)$$

By definition of $X$, the integral in the right-hand side can be written as

$$\int_t^T (T-u) \, dX_u^i = -\kappa_i \int_t^T (T-u) \, X_u^i \, du + \sum_{j=1}^N \rho_{ij} \int_t^T (T-u) \, dW_u^j.$$

Note also that

$$\int_t^T (T-u) \, X_u^i \, du =$$

$$= X_t^i \int_t^T (T-u) \, e^{-\kappa_i (u-t)} \, du + \sum_{j=1}^N \rho_{ij} \int_t^T (T-u) \int_t^u e^{-\kappa_i (u-s)} \, dW_s^j \, du.$$

Calculating separately the last two integrals, the following result holds

$$\int_t^T (T-u) \, e^{-\kappa_i (u-t)} \, du = \left( \frac{T-t}{\kappa_i} + \frac{e^{-\kappa_i (u-t)} - 1}{\kappa_i^2} \right)$$

25In this appendix we drop the superscript $Q$ and denote the $N$-dimensional Brownian motion $W^Q$ simply by $W$. 

22
and, again by integration by parts,

\[
\int_t^T (T - u) \int_t^u e^{-\kappa_i(u-s)}dW_s^j du = \\
= \int_t^T \left( \int_t^u e^{\kappa_i s}dW_s^j \right) du \left( \int_t^u (T - v) e^{-\kappa_i v}dv \right) = \\
= \left( \int_t^T e^{\kappa_i u}dW_u^j \right) \left( \int_t^T (T - v) e^{-\kappa_i v}dv \right) - \\
- \int_t^T \left( \int_t^u (T - v) e^{-\kappa_i v}dv \right) e^{\kappa_i u}dW_u^j = \\
= \int_t^T \left( \int_t^u (T - v) e^{-\kappa_i v}dv \right) e^{\kappa_i u}dW_u^j = \\
\frac{1}{\kappa_i} \int_t^T \left( T - u + \frac{e^{-\kappa_i(T-u)} - 1}{\kappa_i} \right) dW_u^j.
\]

Substituting the previous terms in (22), the following result holds

\[
\int_t^T X_u^i du = (T - t) X_t^i - \\
-\kappa_i \left[ X_t^i \left( \frac{T - t}{\kappa_i} + \frac{e^{-\kappa_i(T-t)} - 1}{\kappa_i^2} \right) + \sum_{j=1}^N \frac{\rho_{ij}}{\kappa_i} \int_t^T \left( T - u + \frac{e^{-\kappa_i(T-u)} - 1}{\kappa_i} \right) dW_u^j \right] + \\
+ \sum_{j=1}^N \rho_{ij} \int_t^T (T - u) dW_u^j = \\
= \frac{1-e^{-\kappa_i(T-t)}}{\kappa_i} X_t^i + \frac{1}{\kappa_i} \sum_{j=1}^N \rho_{ij} \int_t^T (1 - e^{-\kappa_i(T-u)}) dW_u^j,
\]

that is,

\[
\int_t^T X_u^i du = \frac{1-e^{-\kappa_i(T-u)}}{\kappa_i} X_t^i + \frac{1}{\kappa_i} \sum_{j=1}^N \rho_{ij} \int_t^T (1 - e^{-\kappa_i(T-u)}) dW_u^j. \tag{23}
\]

Then \( y(t, T) = \phi_0 (T - t) + \sum_{i=1}^N \int_t^T X_u^i du \) conditional on \( \mathcal{F}_t \) is normally distributed (see Duffie, 2001) with mean

\[
M(t, T) = \phi_0 (T - t) + \sum_{i=1}^N \frac{1-e^{-\kappa_i(T-t)}}{\kappa_i} X_t^i, \tag{24}
\]

23
where the fact that the stochastic integral in (23) is a martingale was used. The variance of \(y(t,T)\) is

\[
V(t,T) = \text{var}^Q \left[ \sum_{i=1}^{N} \frac{Y_i}{\kappa_i} |\mathcal{F}_t \right],
\]  

(25)

where \(Y_i = \sum_{j=1}^{N} \rho_{ij} \int_{t}^{T} (1 - e^{-\kappa_i(T-u)}) \, dW^j_u \). Then

\[
V(t,T) = \sum_{i=1}^{N} \frac{\text{var}^Q (Y_i | \mathcal{F}_t)}{\kappa_i^2} + 2 \sum_{i=1}^{N} \sum_{k>i} \text{cov}^Q (Y_i, Y_k | \mathcal{F}_t) \kappa_i \kappa_k.
\]

(26)

Using Ito's isometry

\[
V(t,T) = \sum_{i=1}^{N} \frac{1}{\kappa_i} \sum_{j=1}^{N} \rho_{ij}^2 \int_{t}^{T} (1 - e^{-\kappa_i(T-u)})^2 \, du +
\]

\[
+ 2 \sum_{i=1}^{N} \sum_{k>i} \frac{1}{\kappa_i \kappa_k} \sum_{j=1}^{N} \rho_{ij} \rho_{kj} \int_{t}^{T} (1 - e^{-\kappa_i(T-u)}) (1 - e^{-\kappa_k(T-u)}) \, du.
\]

(26)

At this point, simple integration produces

\[
V(t,T) = \sum_{i=1}^{N} \frac{1}{\kappa_i} \left( \tau + \frac{2}{\kappa_i} e^{-\kappa_i \tau} - \frac{1}{2\kappa_i} e^{-2\kappa_i \tau} - \frac{3}{2\kappa_i} \right) \sum_{j=1}^{N} \rho_{ij}^2 +
\]

\[
+ 2 \sum_{i=1}^{N} \sum_{k>i} \frac{1}{\kappa_i \kappa_k} \left( \tau + \frac{e^{-\kappa_i \tau} - 1}{\kappa_i} + \frac{e^{-\kappa_k \tau} - 1}{\kappa_k} - \frac{e^{-(\kappa_i + \kappa_k) \tau} - 1}{\kappa_i + \kappa_k} \right) \sum_{j=1}^{N} \rho_{ij} \rho_{kj},
\]

(27)

where \(\tau = T - t\).

**Proof. Lemma 2**

The martingale condition for bond prices (Duffie, 2001) gives:

\[
P(t,T) = \mathbb{E}^Q \left[ e^{-\int_{t}^{T} r_u \, du} | \mathcal{F}_t \right] = \mathbb{E}^Q \left[ e^{-y(t,T)} | \mathcal{F}_t \right].
\]

(28)

Now the normality of variable \(y(t,T)\) (Lemma 1), and a simple property of the mean of log-normal distributions complete the proof.

**Proof. Lemma 3**

By Equation (11) the proof consists of a simple calculation of the expectation
\[ E^Q \left[ \max (IDI_t - Ke^{-y}, 0) | \mathcal{F}_t \right]. \]

\[ c(t, T) = E^Q \left[ \max (IDI_t - Ke^{-y}, 0) | \mathcal{F}_t \right] = \]

\[ = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \max (IDI_t - Ke^{-y}, 0) e^{-\frac{(y-M(t,T))^2}{2V(t,T)}} dy = \] (29)

\[ = \int_{\log(K/IDI_t)}^{\infty} \frac{1}{\sqrt{2\pi}} (IDI_t - Ke^{-y}) e^{-\frac{(y-M(t,T))^2}{2V(t,T)}} dy. \]

Making the substitution \( z = \frac{y-M(t,T)}{\sqrt{V(t,T)}} \) the following result holds:

\[ c(t, T) = \int_{-d}^{d} \frac{1}{\sqrt{2\pi}} \left( IDI_t - Ke^{-z\sqrt{V(t,T)}-M(t,T)} \right) e^{-\frac{1}{2}z^2} dz = \]

\[ = IDI_t \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz - K \int_{-d}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z\sqrt{V(t,T)}-M(t,T)-\frac{1}{2}z^2} dz = \] (30)

\[ = IDI_t \Phi(d) - Ke^{-M(t,T)+\frac{V(t,T)}{2}} \int_{-d}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z+\sqrt{V(t,T)})^2} dz. \]

where \( d \) is given by Equation (13). Making a new substitution \( v = z + \sqrt{V(t,T)} \) and using Lemma 2 results in (12).
Appendix B

6.1 The Stochastic Volatility Model \((A_1(3))\)

We implement a version of an \(A_1(3)\) model as a robustness check for the results obtained with the Gaussian model. An \(A_1(3)\) model is characterized by the presence of three state variables with one of them driving the conditional volatility of the short-term rate (Dai and Singleton, 2000). Our \(A_1(3)\) model is a particular form of a more general one, where the short-term rate is the sum of one independent CIR process (see Cox et al., 1985) with two Gaussian processes:

\[
   r_t = \phi_0 + X_t + Y_t + Z_t, \tag{31}
\]

with:

\[
   dX_t = \kappa_X(\theta - X_t)dt + \rho_X \sqrt{X_t}dW^Q_X(t), \tag{32}
\]

\[
   dY_t = \kappa_Y(Y_t + \lambda_{Y_0}X_t) \quad \text{and} \quad dW^Q_Y(t) = \rho_{Y}dW^Q_Y(t),
\]

\[
   dW^Q_Z(t) = \rho dW^Q_{YZ}(t), \tag{33}
\]

where \(W^Q_X, W^Q_Y\) and \(W^Q_Z\) are independent Brownian motions under the pricing measure \(Q\), and where we use the following short notation:

\[
   dY_Z_t = \begin{bmatrix} dY_t \\ dZ_t \end{bmatrix} \quad \text{and} \quad dW^Q_{YZ}(t) = \begin{bmatrix} dW^Q_Y(t) \\ dW^Q_Z(t) \end{bmatrix}.
\]

The transition from the pricing measure \(Q\) to the objective probability measure \(P\) is given by the extended affine market prices of risks by Cheridito et al. (2006)\(^{26}\):

\[
   dW^Q_X(t) = dW^P_X(t) + \frac{1}{\sqrt{X_t}} \left( \lambda_{Y_0}^X + \lambda_{Y_1}^X X_t \right) dt \quad \tag{34}
\]

\(^{26}\)In the Gaussian model, this market price of risk coincides with the essentially affine market price of risk (Duffee, 2002), indicating that both models (Gaussian and \(A_1(3)\)) are implemented with the most general market prices that maintain affine dynamics under both risk neutral and objective measures.
and
\[ dW_Q^{YZ}(t) = dW^P_Y(t) + \left( \lambda_0^YZ + \lambda_1^YZ Z_t \right) dt, \quad (35) \]
where \( \lambda_0^X \) and \( \lambda_1^X \) are real numbers, \( \lambda_0^YZ \) is a vector in \( \mathbb{R}^2 \), and \( \lambda_1^YZ \) is a 2 \times 2 matrix.

By the independence of the CIR process, it follows directly from the results of Section 3 and from Brigo and Mercurio (2001) that the time \( t \) price of a zero-coupon bond maturing at time \( T \) is given by (\( \tau = T - t \)):
\[ P(t, T) = e^{A(\tau) + B_X(\tau)X_t + B_Y(\tau)Y_t + B_Z(\tau)Z_t}, \quad (36) \]
where
\[ A(\tau) = -\phi_0 \tau + \frac{2\kappa_X \theta}{\rho_X} \ln \left( \frac{2\gamma e^{\kappa_X + \gamma}}{2\gamma + (\kappa_X + \gamma)(e^{\gamma} - 1)} \right) + \frac{V_{YZ}(t, T)}{2}, \]
\[ B_X(\tau) = -\frac{2(e^{\gamma} - 1)}{2\gamma + (\kappa_X + \gamma)(e^{\gamma} - 1)}, \]
\[ B_Y(\tau) = -\frac{1 - e^{\eta_Y \tau}}{\eta_Y}, \]
and
\[ B_Z(\tau) = -\frac{1 - e^{\eta_Z \tau}}{\eta_Z}, \]
with \( \gamma = \sqrt{\kappa_X^2 + 2\rho_X^2} \) and \( V_{YZ} \) given by (6), with \( N = 2 \) and \( \kappa = \eta \).

According to the explanation in Section 3, the pricing of IDI options demands knowledge of the distribution of \( y(t, T) \) conditional on the information available at time \( t \). If \( r_t \) is a Gaussian process, then \( y|\mathcal{F}_t \) is normally distributed. However, in the \( A_1(3) \) case there is not a simple numerical procedure to calculate probabilities related to \( y \). Nevertheless, the moment of order \( m \) of \( e^{-y(t, T)} \) can be calculated, since it is the price of a bond when the short-term rate is \( m \times r_t \). Therefore we can use an Edgeworth expansion technique to obtain IDI Asian option prices (see Collin-Dufresne and Goldstein, 2002b, for its use on swaptions pricing). In the next lines we describe how to apply this technique in our \( A_1(3) \) version. Using the forward measure approach (Geman et al., 1995), the price of the IDI option is given by:
\[ c(t, T) = IDI_t \mathcal{P}_t^Q(e^{-y} < IDI_t/K) - PK(t, T)\mathcal{P}_t^T(e^{-y} < IDI_t/K), \]
27
where $\mathcal{P}_t$ denotes probabilities conditional on $\mathcal{F}_t$ and the superscript $T$ represents the forward measure.

The probabilities in the right-hand side of the above equation can be calculated using the Edgeworth expansion. The Edgeworth expansion is basically an expansion of a distribution around the normal distribution. Following Collin-Dufresne and Goldstein (2002b), we expand up to the seventh order. In this case, the probabilities $\mathcal{P}_t$ are equal to $1 - \sum_{j=0}^{7} \alpha_j \beta_j$, where the coefficients $\alpha_j$ are ratios of polynomials of order 7 in the moments of $e^{-y(t,T)}$ and $\beta_j$ are simple functions of the cumulative normal function and the first two moments of $e^{-y(t,T)^{27}}$. The moments of $e^{-y(t,T)}$ under both the risk neutral measure $Q$ and the forward measure are obtained as described in the previous paragraph. However, the computation of these moments under the forward measure are slightly more difficult, since $r_t$ under the forward measure follows a Hull and White (1990) process with time-varying parameters. Therefore, we do not have closed-form expressions for bonds, and adopt the Runge-Kutta method to solve numerically the coupled pair of differential equations satisfied by terms $A$ and $B$ appearing in the bond expression.

\footnote{Collin-Dufresne and Goldstein (2002b) provide the precise expressions for the coefficients $\alpha_j$ and $\beta_j$.}
Appendix C - Estimation Procedure

Gaussian Model

In this work, in the Gaussian model the maximum likelihood estimation procedure described in Chen and Scott (1993) is extended to deal with options.

The following bond yields are observed along $H$ different days: $rb_t(1/252)$, $rb_t(21/252)$, $rb_t(63/252)$, $rb_t(126/252)$, $rb_t(189/252)$, $rb_t(1)$ and $rb_t(1.5)$. Let $rb$ represent the $H \times 7$ matrix containing the yields for all $H$ days. In addition, the price $cs_t$ for an at-the-money call with time to maturity $95/252$ years is observed during the same $H$ days. Let $cs$ be the vector of length $H$ that represents these call prices. The ID bonds and the at-the-money IDI call are called reference market instruments. Denote by $rmi = [rb, cs]$ the $H \times 8$ matrix containing the yields and the price of these reference market instruments. Assume that the model parameters are represented by vector $\phi$ and a time unit equal to $\Delta t$. Finally, let $g_i(X_t; t, \phi)$ be the function that maps reference market instrument $i$ into state variables.

Since three factors are adopted to estimate the model, it is assumed that reference market instruments, say $i_1$, $i_2$ and $i_3$, are observed without error. For each fixed $t$, the state vector is obtained through the solution of the following system:

$$
g_{i_1}(X_t; t, \phi) = rmi(t, i_1)$$
$$
g_{i_2}(X_t; t, \phi) = rmi(t, i_2)$$
$$
g_{i_3}(X_t; t, \phi) = rmi(t, i_3).$$

(37)

Reference market instruments $i_4$, $i_5$, $i_6$, $i_7$ and $i_8$, are assumed to be observed with Gaussian uncorrelated errors $u_t$:

$$
rm_i(t, [i_4 \ i_5 \ i_6 \ i_7 \ i_8]) - u_t =
$$
$$
\begin{bmatrix}
  g_{i_4}(X_t; t, \phi) & g_{i_5}(X_t; t, \phi) & g_{i_6}(X_t; t, \phi) & g_{i_7}(X_t; t, \phi) & g_{i_8}(X_t; t, \phi)
\end{bmatrix}.
$$

(38)

For the estimation of more general dynamic term structure models based on joint bond-option data, see for instance, Umantsev (2001), Han (2007) and Almeida et al. (2006), among others.

$rb_t(\tau)$ stands for the time $t$ yield of a bond with time to maturity $\tau$. 

28For the estimation of more general dynamic term structure models based on joint bond-option data, see for instance, Umantsev (2001), Han (2007) and Almeida et al. (2006), among others.

29$rb_t(\tau)$ stands for the time $t$ yield of a bond with time to maturity $\tau$. 

29
The log-likelihood function can be written as

\[
L(\phi, rb) = \sum_{t=2}^{H} \log p(X_t|X_{t-1}; \phi) - \\
- \sum_{t=2}^{H} \log |Jac_t| - \frac{H-1}{2} \log |\Omega| - \frac{1}{2} \sum_{t=2}^{H} u'_t \Omega^{-1} u_t,
\]

(39)

where:

1. \( Jac_t \) is the Jacobian matrix of the transformation defined by (37);

2. \( \Omega \) represents the covariance matrix for \( u_t \), estimated using the sample covariance matrix of the \( u_t \)'s implied by the extracted state vector;

3. \( p(X_t|X_{t-1}; \phi) \) is the transition probability from \( X_{t-1} \) to \( X_t \) under the objective probability measure \( \mathbb{P} \).

The final objective of this procedure is to estimate vector \( \phi \), which maximizes function \( L(\phi, rb) \). In order to avoid possible local minima, several different starting parameter vectors are tested and for each on a search for the optimal point is performed, using Nelder-Mead Simplex algorithm for nonlinear optimization and gradient-based optimization methods.

**A1(3) Model**

In the A1(3) model, we also follow a procedure similar to Chen and Scott (1993) but instead of maximum likelihood, we perform a quasi maximum likelihood (QML) estimation. Although the transition probabilities under the A1(3) model is the product of a non-central chi-square density and a Gaussian density, for stability purposes we decided to implement QML, since we have analytical formulas for the first two conditional moments of the state vector in any affine model. Other than applying QML, the rest of the estimation procedure is precisely as described above for the Gaussian model.
References


Table 1: Descriptive statistics of options.
This table presents descriptive statistics of two different options databases. The first is composed of an at-the-money IDI call with maturity of 95 business days and moneyness of one. The second is composed of the most liquid IDI call on each day. The sample size is 748 business days, covering the period from January 2003 until December 2005.

<table>
<thead>
<tr>
<th>Option</th>
<th>Average price</th>
<th>Price volatility</th>
<th>Average moneyness</th>
<th>Average maturity (business days)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Most liquid</td>
<td>989.22</td>
<td>833.37</td>
<td>0.9951</td>
<td>105.01</td>
</tr>
<tr>
<td>At-the-money</td>
<td>281.30</td>
<td>91.16</td>
<td>1</td>
<td>95</td>
</tr>
</tbody>
</table>

Figure 1: Time series of Brazilian bonds yields.
This figure contains the time series of Brazilian bond yields extracted from ID-futures with maturities of 1 month (21 business day), 6 months (126 business days) and 12 months (252 business days) between January 2003 and December 2005.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Standard Error</th>
<th>Ratio</th>
<th>(\frac{\text{abs(Value)}}{\text{Std Error}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\kappa_1)</td>
<td>6.3435</td>
<td>0.0889</td>
<td></td>
<td>71.34</td>
</tr>
<tr>
<td>(\kappa_2)</td>
<td>1.6082</td>
<td>0.0174</td>
<td></td>
<td>92.47</td>
</tr>
<tr>
<td>(\kappa_3)</td>
<td>0.0003</td>
<td>0.00001</td>
<td></td>
<td>12.65</td>
</tr>
<tr>
<td>(\rho_{11})</td>
<td>0.0919</td>
<td>0.0021</td>
<td></td>
<td>43.07</td>
</tr>
<tr>
<td>(\rho_{21})</td>
<td>-0.0216</td>
<td>0.0034</td>
<td></td>
<td>6.30</td>
</tr>
<tr>
<td>(\rho_{22})</td>
<td>0.0400</td>
<td>0.0010</td>
<td></td>
<td>40.22</td>
</tr>
<tr>
<td>(\rho_{31})</td>
<td>-0.0008</td>
<td>0.0016</td>
<td></td>
<td>0.47</td>
</tr>
<tr>
<td>(\rho_{32})</td>
<td>-0.0192</td>
<td>0.0004</td>
<td></td>
<td>50.85</td>
</tr>
<tr>
<td>(\rho_{33})</td>
<td>0.0112</td>
<td>0.0001</td>
<td></td>
<td>108.49</td>
</tr>
<tr>
<td>(\lambda_X(11))</td>
<td>-329.7170</td>
<td>109.0627</td>
<td></td>
<td>3.02</td>
</tr>
<tr>
<td>(\lambda_X(21))</td>
<td>42.9899</td>
<td>68.3982</td>
<td></td>
<td>0.62</td>
</tr>
<tr>
<td>(\lambda_X(22))</td>
<td>0.5462</td>
<td>12.0799</td>
<td></td>
<td>0.05</td>
</tr>
<tr>
<td>(\lambda_X(31))</td>
<td>-200.4261</td>
<td>39.4736</td>
<td></td>
<td>5.07</td>
</tr>
<tr>
<td>(\lambda_X(32))</td>
<td>258.7188</td>
<td>10.6457</td>
<td></td>
<td>24.30</td>
</tr>
<tr>
<td>(\lambda_X(33))</td>
<td>-75.3815</td>
<td>7.9478</td>
<td></td>
<td>9.48</td>
</tr>
<tr>
<td>(\kappa^p_{11})</td>
<td>36.6308</td>
<td>10.0183</td>
<td></td>
<td>3.65</td>
</tr>
<tr>
<td>(\kappa^p_{12})</td>
<td>-8.8291</td>
<td>0.3803</td>
<td></td>
<td>23.21</td>
</tr>
<tr>
<td>(\kappa^p_{22})</td>
<td>1.5863</td>
<td>0.4826</td>
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<td>3.28</td>
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<tr>
<td>(\kappa^p_{31})</td>
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<td>0.9584</td>
<td></td>
<td>2.93</td>
</tr>
<tr>
<td>(\kappa^p_{32})</td>
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<td>0.1134</td>
<td></td>
<td>25.43</td>
</tr>
<tr>
<td>(\kappa^p_{33})</td>
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<td>0.0889</td>
<td></td>
<td>9.48</td>
</tr>
<tr>
<td>(\phi_0)</td>
<td>0.18</td>
<td>-</td>
<td></td>
<td>-</td>
</tr>
</tbody>
</table>

**Table 2: Gaussian model parameters - Bond version.**

This table presents parameter values and standard errors for the bond version of the Gaussian model. The model was estimated by maximum likelihood, following the method proposed by Chen and Scott (1993). Standard errors were obtained by the BHHH method. Boldface values mean that the parameter is significant at a 95% confidence level. The \(P\) superscripts indicate parameters of the physical dynamics.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Standard Error</th>
<th>Ratio</th>
<th>( \frac{\text{abs(Value)}}{\text{Std Error}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \kappa_1 )</td>
<td>37.6296</td>
<td>10.8910</td>
<td>3.46</td>
<td></td>
</tr>
<tr>
<td>( \kappa_2 )</td>
<td>3.4565</td>
<td>0.1858</td>
<td>18.60</td>
<td></td>
</tr>
<tr>
<td>( \kappa_3 )</td>
<td>0.0003</td>
<td>0.00002</td>
<td>16.96</td>
<td></td>
</tr>
<tr>
<td>( \rho_{11} )</td>
<td>0.0951</td>
<td>0.0040</td>
<td>23.77</td>
<td></td>
</tr>
<tr>
<td>( \rho_{21} )</td>
<td>-0.0415</td>
<td>0.0044</td>
<td>9.41</td>
<td></td>
</tr>
<tr>
<td>( \rho_{22} )</td>
<td>0.0729</td>
<td>0.0016</td>
<td>45.45</td>
<td></td>
</tr>
<tr>
<td>( \rho_{31} )</td>
<td>-0.0006</td>
<td>0.0017</td>
<td>0.39</td>
<td></td>
</tr>
<tr>
<td>( \rho_{32} )</td>
<td>-0.0332</td>
<td>0.0016</td>
<td>20.72</td>
<td></td>
</tr>
<tr>
<td>( \rho_{33} )</td>
<td>0.0194</td>
<td>0.0003</td>
<td>69.66</td>
<td></td>
</tr>
<tr>
<td>( \lambda_X^{(11)} )</td>
<td>-240.0116</td>
<td>129.1894</td>
<td>1.86</td>
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</tr>
<tr>
<td>( \lambda_X^{(21)} )</td>
<td>-137.1462</td>
<td>63.9335</td>
<td>2.15</td>
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<tr>
<td>( \lambda_X^{(22)} )</td>
<td>0.0376</td>
<td>12.4838</td>
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<td>( \lambda_X^{(31)} )</td>
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<td>84.7153</td>
<td>3.07</td>
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<tr>
<td>( \lambda_X^{(32)} )</td>
<td>16.917</td>
<td>26.6624</td>
<td>0.63</td>
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<tr>
<td>( \lambda_X^{(33)} )</td>
<td>-278.9916</td>
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<tr>
<td>( \kappa_{11}^{\mathbb{P}} )</td>
<td>60.4487</td>
<td>22.8191</td>
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<tr>
<td>( \kappa_{12}^{\mathbb{P}} )</td>
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<td>1.01</td>
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</tr>
<tr>
<td>( \kappa_{22}^{\mathbb{P}} )</td>
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<td>0.0872</td>
<td>39.60</td>
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</tr>
<tr>
<td>( \kappa_{31}^{\mathbb{P}} )</td>
<td>0.3261</td>
<td>0.3153</td>
<td>1.03</td>
<td></td>
</tr>
<tr>
<td>( \kappa_{32}^{\mathbb{P}} )</td>
<td>-0.3263</td>
<td>0.1294</td>
<td>2.52</td>
<td></td>
</tr>
<tr>
<td>( \kappa_{33}^{\mathbb{P}} )</td>
<td>5.4016</td>
<td>2.6102</td>
<td>2.07</td>
<td></td>
</tr>
<tr>
<td>( \phi_0 )</td>
<td>0.18</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>

**Table 3: Gaussian model parameters - Option version.**

This table presents parameter values and standard errors for the option version of the Gaussian model. The model was estimated by maximum likelihood following the method proposed by Chen and Scott (1993). Standard errors were obtained by the BHHH method. Boldface values mean that the parameter is significant at a 95% confidence level. The \( \mathbb{P} \) superscripts indicate parameters of the physical dynamics.
<table>
<thead>
<tr>
<th>Variable</th>
<th>Gaussian model</th>
<th></th>
<th></th>
<th>A₁(3) model</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Bond version</td>
<td>Option version</td>
<td>Bond version</td>
<td>Option version</td>
<td>Bond version</td>
<td>Option version</td>
</tr>
<tr>
<td></td>
<td>Error</td>
<td>Std</td>
<td>Error</td>
<td>Std</td>
<td>Error</td>
<td>Std</td>
</tr>
<tr>
<td>21-day yield</td>
<td>18.10</td>
<td>24.52</td>
<td>29.72</td>
<td>35.37</td>
<td>18.34</td>
<td>24.58</td>
</tr>
<tr>
<td>63-day yield</td>
<td>6.93</td>
<td>9.52</td>
<td>14.89</td>
<td>17.70</td>
<td>7.21</td>
<td>9.62</td>
</tr>
<tr>
<td>126-day yield</td>
<td>-</td>
<td>-</td>
<td>Not used</td>
<td>Not used</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>189-day yield</td>
<td>1.76</td>
<td>2.26</td>
<td>-</td>
<td>-</td>
<td>1.86</td>
<td>2.29</td>
</tr>
<tr>
<td>252-day yield</td>
<td>-</td>
<td>-</td>
<td>12.93</td>
<td>15.92</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>378-day yield</td>
<td>11.52</td>
<td>14.04</td>
<td>39.03</td>
<td>46.54</td>
<td>11.75</td>
<td>14.07</td>
</tr>
<tr>
<td>Level</td>
<td>11.01</td>
<td>15.03</td>
<td>26.62</td>
<td>34.78</td>
<td>11.38</td>
<td>15.17</td>
</tr>
<tr>
<td>Slope</td>
<td>9.23</td>
<td>12.54</td>
<td>41.23</td>
<td>51.73</td>
<td>8.94</td>
<td>12.30</td>
</tr>
<tr>
<td>Curvature</td>
<td>6.48</td>
<td>8.36</td>
<td>33.79</td>
<td>42.52</td>
<td>6.11</td>
<td>8.15</td>
</tr>
</tbody>
</table>

Table 4: Absolute errors of yields and principal components.
This table presents mean absolute errors (columns “Error”) and standard deviations (columns “Std”) of the yields with maturities of 21-, 63-, 126-, 189-, 252-, 378-business days and the factors (level, slope and curvature) implicit in the bond and option versions of the Gaussian and A₁(3) models.
Figure 2: Average cross-section of yields - Gaussian model.
This figure shows the average observed and model-estimated cross-section of yields in the bond (solid line) and option (dashed line) versions of the Gaussian model. In the bond version, the model is estimated based only on the ID-future market, while in the option version information from the IDI call is also included.
Figure 3: Loadings of the dynamic factors - Gaussian model. This figure contains the loadings of the three dynamic factors (level, slope and curvature) in the bond (solid line) and option (dashed line) versions of the Gaussian model. In the bond version, the model is estimated based only on the ID-future market, while in the option version information from the IDI call is also included.
Figure 4: Loadings of the dynamic factors - $A_1(3)$ model.
This figure contains the loadings of the three dynamic factors (level, slope and curvature) in the bond (solid line) and option (dashed line) versions of the $A_1(3)$ model. In the bond version, the model is estimated based only on the ID-future market, while in the option version information from the IDI call is also included.
Figure 5: Time series of the state variables - Gaussian model. This figure contains the time series of the state variables in the Gaussian model between January 2003 and December 2005. The left-hand panel shows the bond version of the model and the right-hand panel the option version. In the bond version, the model is estimated based only on the ID-future market while in the option version information from the IDI call is also included.
This figure shows examples of cross-section instantaneous expected excess returns in the bond (solid line) and option (dashed line) versions of the Gaussian model. In the bond version, the model is estimated based only on the ID-future market, while in the option version information from the IDI call is also included.
Figure 7: 1-year bond excess return - Gaussian model.
This figure shows the time series of instantaneous expected excess return of the 1-year bond for the bond and option versions in the Gaussian model. In the bond version, the model is estimated based only on the ID-future market, while in the option version information from the IDI call is also included.
Figure 8: Bond risk premium decomposition - Gaussian model.
This figure shows the bond risk premium decomposition for the bond version (solid line) and option version (dashed line) in the Gaussian model. In the bond version, the model is estimated based only on the ID-future market, while in the option version information from the IDI call is also included.
Figure 9: Bond risk premium decomposition - $A_1(3)$ model.
This figure shows the bond risk premium decomposition for the bond version (solid line) and option version (dashed line) in the $A_1(3)$ model. In the bond version, the model is estimated based only on the ID-future market, while in the option version information from the IDI call is also included.
Figure 10: IDI call price - Gaussian model.
This figure shows the observed price of the most liquid IDI call as a linear approximation of the model-estimated price in the bond and option versions of the Gaussian model. In the bond version, the model is estimated based only on the ID-future market, while in the option version information from the IDI call is also included.
Figure 11: IDI call relative price error - Gaussian model. 
This figure shows the model relative error \( \frac{\text{Model price}}{\text{Observed price}} - 1 \) when pricing the most liquid IDI call based in parameters estimated in the bond version (solid line) and option version (dotted line) of the Gaussian model. In the bond version, the model is estimated based only on the ID-future market, while in the option version information from the IDI call is also included.
Figure 12: Hedging portfolio - Gaussian model.
This figure shows the time series of the number of units \( q_i, i = 1, 2, 3 \) of state variables in the hedging portfolio in the bond (left-hand panel) and option (right-hand panel) versions of Gaussian model. In the bond version, the model is estimated based only on the ID-future market, while in the option version information from the IDI call is also included.