Ergodic Markov Equilibrium with Incomplete Markets and Short Sales

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Abstract

This paper considers a class of infinite-horizon recursive economies with incomplete markets in which short sales are restricted by an explicit debt ceiling that never binds. I prove existence of an ergodic Markov equilibrium by extending the construction in Duffie, Geanakoplos, Mas-Colell, and McLennan (1994) to economies with short sales. This extension holds under the assumption that agents' marginal utilities are bounded. This particularity is due to the fact that consumption need not be bounded away from zero when heterogeneous agents can accumulate debts over time.

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1 Introduction

In an influential work, Magill and Quinzii (1994) proved existence of a sequential equilibrium for economies with incomplete financial markets in which debt paths were restricted by either personalized transversality conditions, implicit debt constraints, or an explicit uniform debt ceiling that never binds. They also show that those three equilibrium concepts coincide for a broad class of infinite-horizon exchange economies. Similar equivalence results are found in Hernández and Santos (1996) and in Levine and Zame (1996). Those papers are celebrated because they present different ways to rule out Ponzi schemes without introducing additional market imperfections into the economy. Among those alternatives, models with explicit debt constraints became usual in macroeconomics and finance.

In another seminal paper, Duffie, Geanakoplos, Mas-Colell, and McLennan (1994) proved existence of a spotless stationary Markov equilibrium for a class of recursive economies with incomplete financial markets and *without short sales* of assets. This result is important because some type of stationarity is always required for numerical computation of the equilibrium. The authors also show that if one introduces a weak form of sunspots (to convexify the expectations correspondence), then the stationary Markov equilibrium has an ergodic invariant measure. In other words, in a stationary Markov equilibrium process with initial states drawn from the ergodic invariant measure μ, the time series distribution of the state vector (which includes asset prices) asymptotically converges to μ almost surely. Since ergodicity cannot be empirically tested, this class of existence results is our best justification for using asymptotic theory in financial time series.

This paper considers a general class of recursive economies with incomplete markets and short sales. Like in Magill and Quinzii (1994), these economies present a uniform lower bound on impatience. Therefore, there exists a uniform debt ceiling that never binds in equilibrium. This debt ceiling is imposed on individual budget constraints in order to rule out Ponzi schemes, and the structure in Duffie et al. (1994) is adapted to prove existence of a spotless stationary Markov equilibrium and of a conditionally spotless ergodic Markov equilibrium. The existence results hold under the assumption that agents' marginal utilities are bounded. This particularity is due to the fact that individual consumption may get arbitrarily close to zero even when endowments are uniformly bounded away from zero and the Bernoulli utility functions are unbounded from below or satisfy the Inada conditions (see Beker and Chattopadhyay, 2010).

This contribution is of general interest for economists. The ergodic equilibrium result serves as a theoretical support for a broad literature in finance that combines asymptotic time series and asset pricing equations derived under the assumption of unrestricted short sales (e.g., Cochrane, 2001). Furthermore, the stationary equilibrium result contributes to an ongoing controversy on existence of recursive equilibrium in economies with incomplete markets and debt constraints that never binds. Krebs (2004) showed that, for some economies, market clearing and interior optimality conditions imply individual consumption to lie in a bounded open set in any recursive equilibrium.[1] For those economies, the stationary equilibrium process—which is defined here over a compact set—must be constructed in such a way that maximal and minimal consumption values are not reached almost surely.

The remaining of the work is organized as follows. Section 2 describes a general class of recursive exchange economies, defines the Walrasian equilibrium concept, and derives a uniform debt ceiling that never binds. Next, Section 3 proves existence of two related Markov equilibrium concepts for economies with loose debt constraints. Concluding remarks are presented in Section 4.

2 Model

Consider a pure exchange economy with uncertainty, infinitely many periods $t \in \mathbb{T} \equiv \{0,1,...,\infty\}$, multiple consumption goods $l \in \mathbb{L} \equiv \{1,...,L\}$, and long-lived heterogeneous agents $i \in \mathbb{I} \equiv \{1,...,I\}$. The economy's information structure is represented by a Borel probability space $(\Omega, \mathcal{F}, \nu)$ and a filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$ such that $\mathcal{F}_t \subset \mathcal{F}_{t+1} \subset \mathcal{F}$, $\forall t \in \mathbb{T}$. Each node $\omega \in \Omega$ determines a sequence of exogenous shocks $\{s_t(\omega)\}_{t=0}^{\infty}$. It is assumed that $s_t(\omega)$ belongs to a finite set $\mathbb{S} \equiv \{1,...,S\}$ and follows a time-homogeneous Markov process with transition $P(s_{t+1} \mid s_t) \equiv \nu(\{\omega \in \Omega : (s_t(\omega), s_{t+1}(\omega)) = (s_t, s_{t+1})\}) > 0$, for every s_t and s_{t+1} in $\mathbb{S}$.[2]

[1] This result is valid, for instance, for a recursive economy without aggregate risk and populated by two agents with utility functions that are unbounded from below.

[2] The economy's information structure accomodates the possibility of sunspots in equilibrium. It reduces to the usual spotless sequential event tree when $\Omega \equiv \mathbb{S}^{\infty}$ and, for each $t \in \mathbb{T}$, $\mathcal{F}_t$ is the Borel σ-algebra generated by the intersection of all Borel σ-algebras containing the subsets $\{\omega \in \Omega : (s_0(\omega),...,s_t(\omega)) = s^t\}$, for all s^t in the t-fold Cartesian product of $\mathbb{S}$.

For notational convenience, the decision node index (t, ω) is omitted throughout the paper. The time index t is used when the context requires. Otherwise, the subscripts $+1$ and -1 respectively indicate the next-period and previous-period realization of the underlying random variable.

In each period t, individual endowments are determined by time-invariant functions $e_i : \mathbb{S} \to \mathbb{R}^L_{++}$, $\forall i \in \mathbb{I}$. Agent i's preference is numerically represented by a time-separable expected utility function V_i. For any $\mathbb{R}^L_+$-valued stochastic process $\mathbf{x}_i \equiv \{x_{i,t}\}_{t=0}^{\infty}$ on $(\Omega, \mathcal{F}, \nu)$, define:

$$V_i(\mathbf{x}_i) \equiv E_0 \left(\sum_{t \in \mathbb{T}} \beta_i^t u_i(x_{i,t}) \right), \tag{1}$$

where E_0 represents the mathematical expectation conditional on the information set $\mathcal{F}_0$; $\beta_i \in (0, 1)$ is an agent-specific discount factor; and $u_i : \mathbb{D} \to \mathbb{R}$ is a bounded, continuous, nondecreasing, and concave Bernoulli utility function, which is monotonically increasing in its first coordinate and whose domain $\mathbb{D}$ is an open set containing $\mathbb{R}^L_+$.

Remark 1. The assumption that u_i is defined on an open set $\mathbb{D} \supset \mathbb{R}^L_+$ implies that the supergradient set:

$$\partial u_i(x_i) \equiv \{d_i \in \mathbb{R}^L_+ : u_i(\tilde{x}_i) \leq u_i(x_i) + d_i \cdot (\tilde{x}_i - x_i)\,,\ \forall \tilde{x}_i \in \mathbb{D}\}, \tag{2}$$

is nonempty and compact, for every $x_i \in \mathbb{R}^L_+$. Moreover, the supergradient correspondence ∂u_i is upper hemicontinuous on $\mathbb{R}^L_+$. This construction imposes marginal utilities to be bounded on the boundary of $\mathbb{R}^L_+$.[3]

Markets open every period to trade the L consumption goods and $J > 0$ short-lived *numeraire* assets $a_j \in \mathbb{R}^S_+$, $\forall j \in \mathbb{J} \equiv \{1, ..., J\}$. Asset payoffs are linearly

[3] Recursive models usually assume Bernoulli utility functions that are unbounded from below. In many models, this guarantees that individual consumption is bounded away from zero. In those cases, the supergradient correspondence is upper hemicontinuous, nonempty, and compact valued in the subset of the consumption set that is individually rational. This is used, for instance, by Duffie et al. (1994) in a scenario without short sales and by Braido (2008) in an environment with default. However, this technique is not useful in an economy with short sales and without default because—as shown by Beker and Chattopadhyay (2010)—individual consumption may get arbitrarily close to zero even when endowments are uniformly bounded away from zero and the Bernoulli utility functions are unbounded from below or satisfy the Inada conditions.

independent, and there is a risk-free asset. Commodity and asset prices are respectively represented by $p \in \mathbb{R}_+^L$ and $q \in \mathbb{R}_+^J$. In each decision node, these prices lie in $\Delta \equiv \left\{(p,q) \in \mathbb{R}_+^{L+J} : \sum_{l\in\mathbb{L}} p_l + \sum_{j\in\mathbb{J}} q_j = 1\right\}$.

Agents choose their personal consumption bundle and portfolio $(x_i, \theta_i) \in \mathbb{R}_+^L \times \mathbb{R}^J$ taking as given: (i) their previous-period portfolio $\theta_{i,-1} \in \mathbb{R}^J$; (ii) the current prices $(p,q) \in \Delta$; and (iii) the stochastic process describing the future prices. They face a debt ceiling $M > 0$. Their choices $(x_i, \theta_i) \in \mathbb{R}_+^L \times \mathbb{R}^J$ must (almost surely) satisfy the following budget constraints:

$$p \cdot (x_i - e_{i,s}) + q \cdot \theta_i \leq p_1 a_s \cdot \theta_{i,-1}, \tag{3}$$

$$q \cdot \theta_i \geq -M. \tag{4}$$

This economy is characterized by $\mathcal{E} \equiv \{\mathbb{I}, \mathbb{T}, \mathbb{S}, P, (V_i, e_i)_{i\in\mathbb{I}}, a, M\}$. Markets are said to clear when the following feasibility constraint holds (almost surely):

$$\textstyle\sum_{i\in\mathbb{I}} (x_i - e_{i,s}, \theta_i) = \mathbf{0}. \tag{5}$$

2.1 Walrasian Equilibrium

Competitive equilibrium is defined in this setting by a stochastic process $\{z_t\}_{t=0}^{\infty}$ on $(\Omega, \mathcal{F}, \nu)$, where $z \equiv (s, \theta_{-1}, x, \theta, p, q)$ is a state variable composed of the exogenous state $s \in \mathbb{S}$, the previous-period portfolios $\theta_{-1} \in \mathbb{Y}_1 \equiv \mathbb{R}^{JI}$, and the current individual choices and prices $(x, \theta, p, q) \in \mathbb{Y}_2 \equiv \mathbb{R}_+^{LI} \times \mathbb{R}^{JI} \times \Delta$.[4] Let z take value in a nonempty space $\mathbb{Z}$ which embeds the economy's feasibility constraints as follows:

$$\mathbb{Z} \equiv \left\{z \in \mathbb{S} \times \mathbb{Y}_1 \times \mathbb{Y}_2 : \textstyle\sum_{i\in\mathbb{I}} (x_i - e_{i,s}, \theta_i) = \mathbf{0}\right\}. \tag{6}$$

A $\mathbb{Z}$-valued stochastic process $\{z_t\}_{t=0}^{\infty}$ on $(\Omega, \mathcal{F}, \nu)$ is said to be *consistent* for the economy $\mathcal{E}$ if, for each $t \in \mathbb{T}$, z_t is $\mathcal{F}_t$-measurable, the conditional probability of s_{t+1} given $(z_0, ..., z_t)$ is P_{s_t} almost surely, and $\theta_{-1}(z_{t+1}) = \theta(z_t)$ almost surely. For a given consistent process $\{z_t\}_{t=0}^{\infty}$, a policy $\{\tilde{x}_{i,t}, \tilde{\theta}_{i,t}\}_{t=0}^{\infty}$ on $(\Omega, \mathcal{F}, \nu)$ is *budget feasible* if it is $\mathcal{F}_t$-measurable and satisfies (3)-(4) almost surely, $\forall t \in \mathbb{T}$. For a given consistent $\{z_t\}_{t=0}^{\infty}$, a budget-feasible policy $\{\tilde{x}_{i,t}, \tilde{\theta}_{i,t}\}_{t=0}^{\infty}$ is *individually optimal* if there is no other budget-feasible policy $\{x'_{i,t}, \theta'_{i,t}\}_{t=0}^{\infty}$ such that $V_i(\mathbf{x}'_i) > V_i(\tilde{\mathbf{x}}_i)$.

[4]The wealth distribution in this economy is summarized by θ_{-1}. This variable need not be included in the definition of a Walrasian equilibrium, but it will be essential for stationarity.

Definition 1. A *Walrasian equilibrium* is a consistent $\mathbb{Z}$-valued stochastic process $\{z_t\}_{t=0}^{\infty}$ with the property that the policy $\{x_{i,t}, \theta_{i,t}\}_{t=0}^{\infty}$ is individually optimal, $\forall i \in \mathbb{I}$.

2.2 Debt Ceilings that Never Bind

The assumptions on utility functions and endowments guarantee the existence of a uniform lower bound on impatience, in the sense of Magill and Quinzii (1994), Hernández and Santos (1996), and Levine and Zame (1996). In other words, there exists $\rho \in (0,1)$ such that:

$$u_i\left(x_{i,0} + (1,0,...0)\right) + E_0\left(\sum_{t\geq 1} \beta_i^t u_i(\rho x_{i,t})\right) > V_i(\mathbf{x}_i), \tag{7}$$

for any $i \in \mathbb{I}$ and any $\mathbb{R}_+^L$-valued process $\mathbf{x}_i \equiv \{x_{i,t}\}_{t=0}^{\infty}$ on $(\Omega, \mathcal{F}, \nu)$ that is essentially bounded by $\bar{e} \equiv 2\sum_{i\in\mathbb{I}} \max_{s\in\mathbb{S}}(e_{i,s})$.

Lemma 1. Fix some $\rho \in (0,1)$ satisfying equation (7). In any Walrasian equilibrium, the value of individual portfolio is almost surely uniformly bounded by:

$$-\frac{I-1}{1-\rho} \leq q \cdot \theta_i \leq \frac{1}{1-\rho}. \tag{8}$$

Proof. The proof follows from Magill and Quinzii (1994, p. 873). The second inequality holds in equilibrium because, if $q \cdot \theta_i > \frac{1}{1-\rho}$ with positive probability, then agent i would be better off by increasing one unit of current consumption of good 1 in exchange for reducing future consumption and asset holdings to $\rho\left(x_{i,+\tau}, \theta_{i,+\tau}\right)$, $\forall \tau \geq 1$. This portfolio change frees $(1-\rho)\, q \cdot \theta_i$ units of nominal income to be expended in the present. Since $p_1 \leq 1$, this generates income to purchase more than one unit of good 1 in all nodes where $q \cdot \theta_i > \frac{1}{1-\rho}$. Thus, condition (7) closes the argument, since equilibrium consumption bundles are essentially bounded by $\bar{e} \equiv 2\sum_{i\in\mathbb{I}} \max_{s\in\mathbb{S}}(e_{i,s})$. Finally, given that $q \cdot \theta_i \leq \frac{1}{1-\rho}$ almost surely, the feasibility constraints imply the first inequality in (8). ■

Notice that the debt limit $-\frac{I-1}{1-\rho}$ does not depend on the previous-period portfolio or equilibrium future prices. Any explicit debt ceiling $M > \frac{I-1}{1-\rho}$ will never bind in a Walrasian equilibrium.

3 Ergodic Markov Equilibrium

The equilibrium analysis follows the structure formulated in Duffie et al. (1994), where the main elements are the state space and the expectations correspondence. The states $z \equiv (s, \theta_{-1}, x, \theta, p, q) \in \mathbb{Z}$ (see Section 2.1) describes the current endogenous and exogenous variables and is a sufficient statistic for the future evolution of the model. The expectations correspondence $g(z)$ associates each current state $z \in \mathbb{Z}$ to a (possibly empty) set of probability measures over the next-period state variable z_{+1}.

Fix $M > \frac{I-1}{1-\rho}$ and let $\mathcal{B}_{\mathbb{Z}}$ be the Borel σ-algebra over $\mathbb{Z}$, $\mathcal{P}_{\mathbb{Z}}$ be the set of probability measures on $(\mathbb{Z}, \mathcal{B}_{\mathbb{Z}})$, and $g : \mathbb{Z} \to \mathcal{P}_{\mathbb{Z}}$ be a set-valued function defined by conditions (a)-(b).

(a) The set $g(z)$ is empty unless the individual choices embedded in z satisfy the budget-constraint inequalities (3)-(4) and the wealth inequality:

$$p \cdot e_{i,s} + p_1 a_s \cdot \theta_{i,-1} \geq -\frac{I-1}{1-\rho}, \ \forall i \in \mathbb{I}. \tag{9}$$

(b) If $g(z)$ is not restricted to be empty by condition (a), then a measure $\pi \in g(z)$ if and only if:

(b.1) The support of π is the graph of some function mapping $\mathbb{S}$ into $\mathbb{Y}_1 \times \mathbb{Y}_2$.

(b.2) The marginal of π on $\mathbb{S}$ is $P(\cdot \mid s)$ almost surely.

(b.3) The marginal of π on $\mathbb{Y}_1$ is degenerated into θ almost surely.

(b.4) The future states $z_{s_{+1}} \equiv \left(s_{+1}, \theta, x_{s_{+1}}, \theta_{s_{+1}}, p_{s_{+1}}, q_{s_{+1}}\right)$ associated to each of the S mass points in the support of π (indexed by s_{+1}) satisfy the budget-constraint inequalities (3)-(4), for all $i \in \mathbb{I}$.

(b.5) For each $i \in \mathbb{I}$, there exist $(d_i, \lambda_i) \in \partial u_i(x_i) \times \mathbb{R}_+$ and $\left(d_{i,s_{+1}}, \lambda_{i,s_{+1}}\right) \in \partial u_i(x_{i,s_{+1}}) \times \mathbb{R}_+$ associated to each of the S mass points in the support of π (indexed by s_{+1}) such that:

$$d_{i,l} \leq \lambda_i p_l, \ \text{if } x_{i,l} = 0, \ \forall l \in \mathbb{L}; \tag{10}$$

$$d_{i,l} = \lambda_i p_l, \ \text{if } x_{i,l} > 0, \ \forall l \in \mathbb{L}; \tag{11}$$

$$\beta d_{i,l,s_{+1}} P\left(s_{+1} \mid s\right) \leq \lambda_{i,s_{+1}} p_{l,s_{+1}}, \ \text{if } x_{i,l,s_{+1}} = 0, \ \forall \left(l, s_{+1}\right) \in \mathbb{L} \times \mathbb{S}; \tag{12}$$

$$\beta d_{i,l,s_{+1}} P\left(s_{+1} \mid s\right) = \lambda_{i,s_{+1}} p_{l,s_{+1}}, \text{ if } x_{i,l,s_{+1}} > 0, \ \forall \left(l, s_{+1}\right) \in \mathbb{L} \times \mathbb{S}; \quad (13)$$

$$\sum_{s_{+1}=1}^{S} \left[\lambda_{i,s_{+1}} p_{1,s_{+1}} a_{s_{+1}}\right] \leq q\lambda_i, \text{ if } q \cdot \theta_i = -M; \quad (14)$$

$$\sum_{s_{+1}=1}^{S} \left[\lambda_{i,s_{+1}} p_{1,s_{+1}} a_{s_{+1}}\right] = q\lambda_i, \text{ if } q \cdot \theta_i > -M. \quad (15)$$

Condition (a) states that $g(z)$ is empty for states z that are not consistent with budget feasibility and the wealth inequality (9). For now, inequality (9) is just a technical condition which assures that the recursive budget set $\{x_i \in \mathbb{R}_+^L : \exists \theta_i \in \mathbb{R}^J$ $s.t.$ (3)-(4)$\}$ has an interior point, for every $(p, q) \in \Delta$ such that $q \neq \mathbf{0}$. Notice that $q_j > 0$, $\forall j \in \mathbb{J}$, since linearly independent payoff vectors must have at least one positive coordinate and preferences are increasing in good 1.

Condition (b) defines which measures are included in $g(z)$ when this set is not empty. (Notice that $g(z)$ will still be empty if no measure satisfy those conditions.) In words, (b.1) requires all measures in $g(z)$ to have a finite support with S mass points. As a consequence, any transition selected from g will be spotless. Conditions (b.2)-(b.4) require all measures in $g(z)$ to be consistent with the exogenous-shock probability, the law of motion for asset holdings, and the next-period budget constraints.

Finally, condition (b.5) requires that π is such that the endogenous variables in z are dynamically optimal for each agent whose previous debt does not impose zero consumption. This follows from the normal Kuhn-Tucker conditions for nonsmooth concave functions. (See Balder, 2001, and notice that the Slater's condition is satisfied thanks to the wealth inequality in (a), $M > \frac{I-1}{1-\rho}$, and $q \neq \mathbf{0}$.)

Lemma 2. There exists $p_{1,\inf} > 0$ such that, for every state $z \in \mathbb{Z}$ satisfying $g(z) \neq \emptyset$, one has $p_1 \geq p_{1,\inf}$.

Proof. The function u_i is concave, nondecreasing, and increasing in good 1. Moreover, its supergradient set $\partial u_i(x_i)$ is nonempty and bounded, for each $x_i \in \mathbb{R}_+^L$. Define $\delta_{\sup} \equiv \max_{l \in \mathbb{L}} \sup_{(i,x_i) \in \mathbb{I} \times \mathbb{X}} \partial_l u_i(x_i) > 0$ and $\delta_{1,\inf} \equiv \inf_{(i,x_i) \in \mathbb{I} \times \mathbb{X}} \partial_1 u_i(x_i) > 0$, where $\mathbb{X} \equiv \left\{x_i \in \mathbb{R}_+^L : x_i \leq \bar{e}\right\}$ and $\partial_l u_i(x_i)$ is the set composed of the lth coordinates of all vectors in $\partial u_i(x_i)$. Notice also that $\sum_{l \in L} p_l \leq 1$ since $(p, q) \in \Delta$. Moreover, the feasibility condition imposed on $\mathbb{Z}$ implies $x_{i,1} > 0$ for some $i \in \mathbb{I}$. Thus, conditions (10)-(11) in the definition of g imply $p_1 \geq p_{1,\inf} \equiv \frac{\delta_{1,\inf}}{L\delta_{\sup}} > 0$. ■

Next, Lemma 3 shows that the graph of g is closed when states are restricted to lie in a compact subset of $\mathbb{Z}$. This is a key condition for existence of a stationary equilibrium.

Lemma 3. Let K be a compact subset of $\mathbb{Z}$. The correspondence $g(z) \cap \mathcal{P}_K$, for $z \in K$, has a closed graph.

Proof. Take two convergent sequences $\{z_n\}_{n=0}^{\infty} \to z$ and $\{\pi_n\}_{n=0}^{\infty} \to \pi$ such that $z_n \in K$ and $\pi_n \in g(z_n) \cap \mathcal{P}_K$, $\forall n$. To show that $\pi \in g(z) \cap \mathcal{P}_K$, notice the following four facts. First, if z_n satisfies the budget-constraint inequalities (3)-(4) and the wealth inequality (9), for every n, then z satisfies condition (a) in the definition of g. Second, for each n, there exist h_n such that condition (b.1) holds. Since K is compact, $\{h_n\}_{n=0}^{\infty}$ is bounded and, then, it has a subsequence that converges to some function h such that the support of π is the graph of h. Third, for every n, the marginal of π_n on $\mathbb{S}$ is $P(\cdot \mid s)$ and the marginal of π_n on $\mathbb{Y}_1$ is degenerated into θ_n almost surely. Thus, the limit probability π satisfies conditions (b.2)-(b.3)—that is, its marginal on $\mathbb{S}$ and on $\mathbb{Y}_1$ are respectively given by $P(\cdot \mid s)$ and by the Dirac measure at $\theta \equiv \lim \theta_n$. Finally, the conditions in (b.4)-(b.5) are preserved in the limit—since ∂u_i is upper hemicontinuous, nonempty, and compact valued, for $x_i \in \mathbb{R}_+^L$—and then hold for $(z, \pi) \equiv \lim (z_n, \pi_n)$. ■

The economy $\mathcal{E} \equiv \{\mathbb{I}, \mathbb{T}, \mathbb{S}, P, (V_i, e_i)_{i\in\mathbb{I}}, a, M\}$ is represented by the state space $\mathbb{Z}$, the expectations correspondence g, and some arbitrary initial condition $(\hat{s}, \hat{\theta}_{-1}) \in \mathbb{S} \times \mathbb{Y}_1$. Let us now define and prove existence of a first equilibrium concept used in this paper.

Definition 2 (Stationary Markov Equilibrium). A stationary Markov equilibrium for economy $\mathcal{E}$ is a pair (Z, Π), where Z is a measurable subset of $\mathbb{Z}$ and $\Pi : Z \to \mathcal{P}_Z$ is a transitional probability such that $\Pi_z \in g(z)$, $\forall z \in Z$.

A stationary Markov equilibrium is composed of a set of states and a law of motion such that the current realization of z determines the future stochastic equilibrium path. This concept encompass the Walrasian notion of equilibrium, since conditions (a) and (b) in the definition of g imply that any stationary Markov

Z-valued process $\{z_t\}_{t=0}^{\infty}$ with transition Π is consistent and individually optimal. Moreover, since probabilities in g must have S mass points, any stationary Markov equilibrium for g is spotless.

Proposition 1. There exists a (spotless) stationary Markov equilibrium (Z, Π), for any economy $\mathcal{E}$ such that $M > \frac{I-1}{1-\rho}$.

Proof. The proof has three steps. First, let a T-horizon equilibrium be a Walrasian equilibrium for a truncated economy in which time is restricted to lie in $\{0, ..., T\} \subset \mathbb{T}$ and $\theta_{i,T} = \mathbf{0}$, $\forall i \in \mathbb{I}$. For any finite $T > 0$, there exists a spotless T-horizon equilibrium when the initial portfolio is $\hat{\theta}_{-1} = \mathbf{0}$. This result is standard in the literature, and the basic structure of its proof can be found in Geanakoplos and Polemarchakis (1986).

Second, equilibrium consumption bundles are uniformly bounded from above and lie in $\mathbb{X} \equiv \left\{x_i \in \mathbb{R}_+^L : x_i \leq \bar{e}\right\}$ almost surely. Equilibrium portfolios are also uniformly bounded and (almost surely) lie in a compact set $\Theta \subset \mathbb{R}^J$. To see this, notice from (3)-(4) that $p_1 a_s \cdot \theta_{i,-1} \geq -(p \cdot e_{i,s} + M)$, $\forall s$. Thanks to Lemma 2, one can define $\bar{k}_{i,s} \equiv (\mathbf{1} \cdot e_{i,s} + M)/p_{1,\inf} > 0$ such that $a\theta_i^{\perp} \geq -\left(\bar{k}_{i,1}, ..., \bar{k}_{i,S}\right)^{\perp}$, where $a \equiv [a_s]$ is a $S \times J$ matrix and $\perp$ indicates the transpose vector. Since a has full column rank, there exists a $J \times S$ matrix a^{-1} such that $a^{-1}a = I$, where I is the $J \times J$ identity matrix. Therefore, one must have $\theta_i^{\perp} \geq -a^{-1}\left(\bar{k}_{i,1}, ..., \bar{k}_{i,S}\right)^{\perp}$. This uniform lower bound for asset holdings jointly with the market-clearing conditions define a uniform upper bound for θ_i. Therefore, for any finite $T > 0$, there is a T-horizon equilibrium process that is (almost surely) valued in a compact set $K \equiv \mathbb{S} \times \Theta \times \mathbb{X} \times \Theta \times \Delta$.

To conclude the proof, one must find a closed subset of K that is self-justified for g (see Duffie et al., 1994, Proposition 1.1, p. 748). Formally, a self-justified set is a closed set $Z \subset K$ such that $g(z) \cap \mathcal{P}_Z$ is nonempty, for all $z \in Z$. It can be constructed by following the steps used to prove Theorem 1.2 in Duffie et al. (1994). Since that proof is constructive, it is worth presenting a short version here.

Since conditions (a)-(b) in the definition of g hold for any spotless T-horizon equilibrium, the set $g(z)$ is nonempty for some $z \in K$. Let $Z_0 = K$ and $Z_n = \{z \in Z_{n-1} : \exists \pi \in g(z) \;\; s.t. \;\; \sup \pi(Z_{n-1}) = 1\}$, $\forall n > 0$, where the supremum of π is taken over Borel measurable subsets of Z_{n-1}. Define then $Z = \bigcap_{n=0}^{\infty} \bar{Z}_n$, where $\bar{Z}_n$ is the closure of Z_n. Notice that Z is nonempty and compact since it is the intersection of a nested sequence of nonempty compact sets. One can then find a measure $\pi \in g(z) \cap \mathcal{P}_Z$, for each $z \in Z$. Take a sequence $\{z_n\}_{n=0}^{\infty} \to z$ such that $z_n \in Z_n$. For each n, take $\pi_n \in g(z_n)$ such that $\sup \pi(Z_{n-1}) = 1$ (which is possible since $z_n \in Z_n$). Since $\pi_n \in \mathcal{P}_{\bar{Z}_n}$ and $\{\mathcal{P}_{\bar{Z}_n}\}_{n=0}^{\infty}$ is a descending sequence of compact sets with intersection $\mathcal{P}_Z$, there is a measure $\pi \equiv \lim \pi_n$ that lies in $\mathcal{P}_Z$. This limit measure π must lie in $g(z)$ since this correspondence has a closed graph when its domain is restrict to the compact set K (see Lemma 3). This concludes the proof. ∎

It sometimes useful to associate an invariant ergodic measure to a stationary Markov equilibrium transition. This defines a new equilibrium concept which implies that, if the initial state is drawn with distribution μ, then the distribution of all future realization of the system is also μ. This is the analogue of the deterministic notion of a steady state.

Definition 3 (Invariant Ergodic Measure). An invariant ergodic measure for a transition (Z, Π) is a measure $\mu \in \mathcal{P}_Z$ such that: (i) $\mu(\mathcal{A}) \equiv \int_Z \Pi_z(\mathcal{A}) d\mu(z)$, for any measurable set $\mathcal{A} \subset Z$; and (ii) either $\mu(\mathcal{A}) = 1$ or $\mu(\mathcal{A}) = 0$, for any measurable set $\mathcal{A} \subset Z$ such that $\Pi_z \in \mathcal{P}_{\mathcal{A}}$ for μ-a.e. $z \in \mathcal{A}$.

Definition 4 (Ergodic Markov Equilibrium). An ergodic Markov equilibrium for $\mathcal{E}$ is a stationary Markov equilibrium (Z, Π) with an invariant ergodic measure $\mu \in \mathcal{P}_Z$.

Proposition 2. There exists a (conditionally spotless) ergodic Markov equilibrium (Z, Π, μ), for any economy $\mathcal{E}$ such that $M > \frac{I-1}{1-\rho}$.

Proof. Define the expectations correspondence G as the closure of the convex hull of g. The result follows then from Corollary 1.1 (p. 751) and Proposition 1.3 (p. 757) in Duffie et al. (1994). ∎

The term conditionally spotless deserves an explanation. According to Corollary 1.1 in Duffie et al. (1994, p. 751), ergodicity can be obtained when the expectations correspondence is convex-valued. The correspondence g is not convex-valued because of condition (b.1), which rules out sunspots. By taking G as the closure of the convex hull of g, one allows for randomizations over spotless equilibrium transitions (thereby, re-introducing sunspots). As mentioned in Duffie et al. (1994), for $Z \subset \mathbb{S} \times \mathbb{Y}_1 \times \mathbb{Y}_2$, the Skorokhod Theorem allows one to represent any transition

$\Pi : Z \to \mathcal{P}_Z$ in the form $(x_{+1}, \theta_{+1}, p_{+1}, q_{+1}) = f\left(s_{+1}, \eta_{+1}, z\right)$, where f is a function and η_{+1} follows an *i.i.d.* uniform distribution in $[0, 1]$. In our case, the agents' optimality conditions hold (almost surely) for each realization of the sunspot variable $\eta \in [0, 1]$. This is as if agents observed the sunspot variable before making their decisions in each node. Thus, the ergodic Markov equilibrium is spotless conditional on the realization of η.

4 Concluding Remarks

A few concluding remarks must be made. First, differently from Duffie et al. (1994), the existence results in this paper do not hold for any initial portfolio $\hat{\theta}_{-1}$. In economies with short sales, individual initial debts cannot be arbitrary since, otherwise, the set of budget-feasible allocations might be empty. In the proof of Proposition 1, an equilibrium is shown to exist when the initial portfolio is $\hat{\theta}_{-1} = \mathbf{0}$. If financial trades occur in equilibrium (which is typically the case), then there are nonnull portfolios that could be drawn in the initial state (for a given equilibrium process). In fact, any state drawn from the ergodic invariant measure μ can be an initial condition. The only particularity to be noticed is that the set of possible initial conditions is endogenous and depends on the stochastic process describing the equilibrium prices. The choice $\hat{\theta}_{-1} = \mathbf{0}$ is the simplest available for computation. However, there is no guarantee that the stationary equilibrium process will converge to an invariant measure μ unless the process is initially drawn from μ.

A second issue to be stressed is that, for economies with $M > \frac{I-1}{1-\rho}$, the debt constraints will never bind in equilibrium. Therefore, asset prices are given by:

$$q = \sum_{s_{+1}=1}^{S} \left[\frac{\lambda_{i,s_{+1}}}{\lambda_i} p_{1,s_{+1}} a_{s_{+1}} \right], \tag{16}$$

where λ_i and $\lambda_{i,s_{+1}}$ are the Kuhn-Tucker multipliers defined in Section 3—see condition (b.5) in the definition of g.

Marginal utilities were assumed to be bounded in the frontier of $\mathbb{R}_{+}^{L}$ in order to guarantee the closed-graph property of g (see Lemma 3). Nevertheless, the model still admits cases in which $x_{i,1} > 0$ almost surely, $\forall i \in \mathbb{I}$. In those cases, and assuming that u_i is differentiable, the asset pricing equation reduces to:

$$\frac{q}{p_1} = \beta \sum_{s_{+1}=1}^{S} P\left(s_{+1} \mid s\right) \frac{\partial u_i\left(x_{i,s_{+1}}\right)/\partial x_{i,1,s_{+1}}}{\partial u_i\left(x_{i,s}\right)/\partial x_{i,1,s}} a_{s_{+1}}. \tag{17}$$

Krebs (2004) showed that, for some economies in this context, no maximum or minimum consumption level can be achieved with positive probability. For those economies, even though the equilibrium state variable takes values in a compact self-justified set, the equilibrium transition will assure that maximum and minimal consumption values are not reached almost surely.

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