Appendix to

Money Supply and Capital Accumulation on the Transition Path Revisited,

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Appendix A

We wish to show that D's characteristic polynomial is (9) indeed. One has

$$\begin{aligned} & -\lambda^3 + (\operatorname{tr} D)\,\lambda^2 - (D_{11}D_{22} - D_{12}D_{21} + D_{11}D_{33} - D_{13}D_{31} + D_{22}D_{33} - D_{23}D_{32})\,\lambda + \det D \\ = \; & -\lambda^3 + (\operatorname{tr} D)\,\lambda^2 - (D_{11}D_{33} - D_{13}D_{31} + D_{22}D_{33})\,\lambda + \det D, \end{aligned}$$

and

$$\begin{aligned} \operatorname{tr} D &= \delta + \eta, \\ D_{11}D_{33} - D_{13}D_{31} + D_{22}D_{33} &= D_{33}\,(D_{11} + D_{22}) - D_{13}D_{31} = \delta\eta - f''\frac{u_1 + mu_{12}}{u_{11}}, \\ \det D &= -mf''J_2\left(m\frac{u_{12}}{u_{11}} - \frac{u_1 + mu_{12}}{u_{11}}\right) = mf''J_2\frac{u_1}{u_{11}}, \end{aligned}$$

where $\eta := m\,(J_1 u_{12}/u_{11} - J_2)$, and the determinant was calculated by application of Laplace's rule to D's third row. So the characteristic polynomial is

$$-\lambda^3 + (\delta + \eta)\,\lambda^2 - \left(\delta\eta - f''\frac{u_1 + mu_{12}}{u_{11}}\right)\lambda + mf''J_2\frac{u_1}{u_{11}},$$

and it is easy to compare coefficient by coefficient to see that this expression coincides with (9).

Appendix B

Here we check that the assumptions made on u at the beginning of Section 2 are satisfied by the functional form (13). Let $v : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ be given by $v\,(c, m) = c\varphi\,(m/c)$, so that $u = g_\sigma \circ v$.

Function g_σ is obviously increasing and concave: for any $x \in \mathbb{R}_{++}$, $g'_\sigma\,(x) = x^{-\sigma} > 0$ and $g''_\sigma\,(x) = -\sigma x^{-\sigma-1} < 0$. As for v, let $(c, m) \in \mathbb{R}^2_{++}$ and $z := m/c$. Then $v_1\,(c, m) = \varphi\,(z) - z\varphi'\,(z)$, which is positive (the strict concavity and nonnegativity of φ give $\varphi'(z)(0 - z) > \varphi(0) - \varphi(z) \geq -\varphi\,(z)$) and $v_2\,(c, m) = \varphi'\,(z) > 0$. Using that $z_1 = -z/c$, we also have $v_{11}\,(c, m) = (\varphi'\,(z) - \varphi'\,(z) - z\varphi''\,(z))\,z_1 = \left(z^2/c\right)\varphi''\,(z) < 0$, $v_{22}\,(c, m) = (1/c)\,\varphi''\,(z) < 0$, $v_{12}\,(c, m) = v_{21}\,(c, m) = (-z/c)\,\varphi''\,(z)$ and $\left(v_{11}v_{22} - v_{12}^2\right)(c, m) = 0$, so that v is also concave. Therefore, u is concave.

Although many of the following calculations wouldn't have to be carried through since it is only the sign of these expressions that matters for now, we do it since these expressions shall be needed in Appendix D. All the derivatives of v below are evaluated at (c,m), and all the derivatives of g_σ at $v(c,m)$:

$$\begin{aligned}
u_1 &= g'_\sigma v_1 > 0 \\
&= (c\varphi(z))^{-\sigma}(\varphi(z) - z\varphi'(z)), \\
u_2 &= g'_\sigma v_2 > 0 \\
&= (c\varphi(z))^{-\sigma}\varphi'(z), \\
u_{11} &= g''_\sigma v_1^2 + g'_\sigma v_{11} < 0 \\
&= (c\varphi(z))^{-\sigma-1}\left(-\sigma(\varphi(z) - z\varphi'(z))^2 + z^2\varphi(z)\varphi''(z)\right), \\
u_{22} &= g''_\sigma v_2^2 + g'_\sigma v_{22} < 0 \\
&= (c\varphi(z))^{-\sigma-1}\left(-\sigma\varphi'(z)^2 + \varphi(z)\varphi''(z)\right), \\
u_{12} &= g''_\sigma v_1 v_2 + g'_\sigma v_{12} = (c\varphi(z))^{-\sigma-1}(-\sigma\varphi'(z)(\varphi(z) - z\varphi'(z)) - z\varphi(z)\varphi''(z)).
\end{aligned}$$

Also, for $i \in \{1,2\}$,

$$J_i = \left(\frac{u_2}{u_1}\right)_i = \frac{d}{dz}\left(\frac{\varphi'(z)}{\varphi(z) - z\varphi'(z)}\right) z_i,$$

so that

$$\begin{aligned}
J_1 &= \frac{\varphi''(z)(\varphi(z) - z\varphi'(z)) - \varphi'(z)(-z\varphi''(z))}{(\varphi(z) - z\varphi'(z))^2}\left(-\frac{z}{c}\right) = -\frac{z}{c}\frac{\varphi(z)\varphi''(z)}{(\varphi(z) - z\varphi'(z))^2} > 0, \\
J_2 &= \frac{1}{c}\frac{\varphi(z)\varphi''(z)}{(\varphi(z) - z\varphi'(z))^2} = -\frac{J_1}{z} < 0.
\end{aligned}$$

Appendix C

Here we confirm that the instantaneous utility functions taken in Fischer (1979) and Asako (1983) are limiting cases of ours. More specifically, we wish to check that the function $v : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ given by $v(c,m) = \left(\rho c^{\frac{\alpha-1}{\alpha}} + (1-\rho)(\gamma m)^{\frac{\alpha-1}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}}$ is such that $\lim_{\alpha\to1} v(c,m) = c^\rho(\gamma m)^{1-\rho}$ and $\lim_{\alpha\to 0_+} v(c,m) = \min(c,\gamma m)$. In fact,

$$\begin{aligned}
\lim_{\alpha\to1} v(c,m) &= \exp\lim_{\alpha\to1}\frac{\log\left(\rho c^{\frac{\alpha-1}{\alpha}} + (1-\rho)(\gamma m)^{\frac{\alpha-1}{\alpha}}\right)}{\frac{\alpha-1}{\alpha}} \\
&= \exp\lim_{\alpha\to1}\frac{\left(\rho c^{\frac{\alpha-1}{\alpha}}\log c + (1-\rho)(\gamma m)^{\frac{\alpha-1}{\alpha}}\log(\gamma m)\right)\frac{d}{d\alpha}\left(\frac{\alpha-1}{\alpha}\right)}{\left(\rho c^{\frac{\alpha-1}{\alpha}} + (1-\rho)(\gamma m)^{\frac{\alpha-1}{\alpha}}\right)\frac{d}{d\alpha}\left(\frac{\alpha-1}{\alpha}\right)} \\
&= \exp(\rho\log c + (1-\rho)\log(\gamma m)) = c^\rho(\gamma m)^{1-\rho},
\end{aligned}$$

where the second equality used l'Hôpital's rule, and

$$\begin{aligned}
\lim_{\alpha\to0_+} v(c,m) &= \lim_{n\to+\infty}\left(\rho c^{-n} + (1-\rho)(\gamma m)^{-n}\right)^{-\frac{1}{n}} \\
&= \lim_{n\to+\infty}\frac{1}{\left(\rho\left(\frac{1}{c}\right)^n + (1-\rho)\left(\frac{1}{\gamma m}\right)^n\right)^{\frac{1}{n}}} \\
&= \frac{1}{\max\left(\frac{1}{c},\frac{1}{\gamma m}\right)} = \min(c,\gamma m),
\end{aligned}$$

where the third equality used the fact that, if $0 \leq a \leq b$ and $x_n := (\rho a^n + (1-\rho) b^n)^{\frac{1}{n}}$, then $\lim_{n\to+\infty} x_n = b$, since $x_n \leq (\rho b^n + (1-\rho) b^n)^{\frac{1}{n}} = b$ and $\lim_{n\to+\infty} x_n \geq \lim_{n\to+\infty} ((1-\rho) b^n)^{\frac{1}{n}} = b$.

Appendix D

Here we work out the derivatives of u given by (15) that are necessary for the derivation of r in (17), the coefficient of relative risk aversion σ_R in (16), and Ψ in (18) and (19). Given the developments made in Appendix B, we can forget about u and focus on φ only. As mentioned in Section 2, we should use $\varphi(z) = \left(\rho + (1-\rho)(\gamma z)^{\frac{\alpha-1}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}}$. Therefore $\varphi'(z) = \left(\rho + (1-\rho)(\gamma z)^{\frac{\alpha-1}{\alpha}}\right)^{\frac{1}{\alpha-1}} (1-\rho)\gamma^{\frac{\alpha-1}{\alpha}} z^{-\frac{1}{\alpha}} = (1-\rho)\gamma^{\frac{\alpha-1}{\alpha}} (\varphi(z)/z)^{\frac{1}{\alpha}}$, $\varphi(z) - z\varphi'(z) = \varphi(z)^{\frac{1}{\alpha}} \times \left(\varphi(z)^{\frac{\alpha-1}{\alpha}} - (1-\rho)\gamma^{\frac{\alpha-1}{\alpha}} z^{1-\frac{1}{\alpha}}\right) = \rho\varphi(z)^{\frac{1}{\alpha}}$, so that $\varphi''(z) = (1/\alpha)(1-\rho)\gamma^{\frac{\alpha-1}{\alpha}} (\varphi(z)/z)^{\frac{1}{\alpha}-1} \times \left(-(\varphi(z) - z\varphi'(z))/z^2\right) = -(1/\alpha)\rho(1-\rho)\gamma^{\frac{\alpha-1}{\alpha}} z^{-1-\frac{1}{\alpha}} \varphi(z)^{\frac{2-\alpha}{\alpha}}$.

Using the expressions found in the previous appendix,

$$\begin{aligned}
u_1 &= \rho (c\varphi(z))^{-\sigma} \varphi(z)^{\frac{1}{\alpha}}, \\
u_2 &= (1-\rho)\gamma^{\frac{\alpha-1}{\alpha}} (c\varphi(z))^{-\sigma} z^{-\frac{1}{\alpha}} \varphi(z)^{\frac{1}{\alpha}} \\
u_{11} &= (c\varphi(z))^{-\sigma-1} \left(-\sigma\rho^2 \varphi(z)^{\frac{2}{\alpha}} - \frac{\rho(1-\rho)}{\alpha}\gamma^{\frac{\alpha-1}{\alpha}} z^{1-\frac{1}{\alpha}} \varphi(z)^{\frac{2}{\alpha}}\right) \\
&= -\frac{\rho}{\alpha} (c\varphi(z))^{-\sigma-1} \varphi(z)^{\frac{2}{\alpha}} \left(\alpha\sigma\rho + (1-\rho)(\gamma z)^{\frac{\alpha-1}{\alpha}}\right), \\
u_{12} &= (c\varphi(z))^{-\sigma-1} \left(-\sigma\rho(1-\rho)\gamma^{\frac{\alpha-1}{\alpha}} z^{-\frac{1}{\alpha}} \varphi(z)^{\frac{2}{\alpha}} + \frac{\rho(1-\rho)}{\alpha}\gamma^{\frac{\alpha-1}{\alpha}} z^{-\frac{1}{\alpha}} \varphi(z)^{\frac{2}{\alpha}}\right) \\
&= \frac{\rho(1-\rho)(1-\alpha\sigma)\gamma^{\frac{\alpha-1}{\alpha}}}{\alpha} (c\varphi(z))^{-\sigma-1} z^{-\frac{1}{\alpha}} \varphi(z)^{\frac{2}{\alpha}}, \\
J_1 &= -\frac{z}{c}\frac{\varphi(z)\varphi''(z)}{(\varphi(z) - z\varphi'(z))^2} = \frac{1}{c}\frac{\frac{\rho(1-\rho)}{\alpha}\gamma^{\frac{\alpha-1}{\alpha}} z^{-\frac{1}{\alpha}} \varphi(z)^{\frac{2}{\alpha}}}{\rho^2 \varphi(z)^{\frac{2}{\alpha}}} = \frac{(1-\rho)\gamma^{\frac{\alpha-1}{\alpha}}}{\rho\alpha}\frac{z^{-\frac{1}{\alpha}}}{c}, \\
J_2 &= -\frac{J_1}{z} = -\frac{(1-\rho)\gamma^{\frac{\alpha-1}{\alpha}}}{\rho\alpha}\frac{z^{-1-\frac{1}{\alpha}}}{c}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{u_2}{u_1} &= \frac{(1-\rho)\gamma^{\frac{\alpha-1}{\alpha}} z^{-\frac{1}{\alpha}}}{\rho}, \\
\frac{u_1}{u_{11}} &= -\alpha\frac{c\varphi(z)}{\varphi(z)^{\frac{1}{\alpha}}\left(\alpha\sigma\rho + (1-\rho)\gamma^{\frac{\alpha-1}{\alpha}} z^{\frac{\alpha-1}{\alpha}}\right)} = -\alpha c\frac{\rho + (1-\rho)(\gamma z)^{\frac{\alpha-1}{\alpha}}}{\alpha\sigma\rho + (1-\rho)(\gamma z)^{\frac{\alpha-1}{\alpha}}},
\end{aligned}$$

so that (and now using $K^{\frac{1}{\alpha}} = ((1-\rho)/\rho)\gamma^{\frac{\alpha-1}{\alpha}}$)

$$\begin{aligned}
r &:= \frac{u_2}{u_1} = K^{\frac{1}{\alpha}} z^{-\frac{1}{\alpha}} = \left(K\frac{c}{m}\right)^{\frac{1}{\alpha}}, \\
\sigma_R &:= -c\frac{u_{11}}{u_1} = \frac{1}{\alpha}\frac{\alpha\sigma\rho + (1-\rho)(\gamma z)^{\frac{\alpha-1}{\alpha}}}{\rho + (1-\rho)(\gamma z)^{\frac{\alpha-1}{\alpha}}} = \frac{c^{\frac{\alpha-1}{\alpha}}\sigma + K^{\frac{1}{\alpha}} m^{\frac{\alpha-1}{\alpha}}\frac{1}{\alpha}}{c^{\frac{\alpha-1}{\alpha}} + K^{\frac{1}{\alpha}} m^{\frac{\alpha-1}{\alpha}}}.
\end{aligned}$$

Also, at the steady state (in order to find the characteristic polynomial), we have (where c and m are short for

c^* and m^*)

$$\begin{aligned}
\frac{u_1}{u_{11}} &= -\alpha c\frac{1+K^{\frac{1}{\alpha}}z^{\frac{\alpha-1}{\alpha}}}{\alpha\sigma+K^{\frac{1}{\alpha}}z^{\frac{\alpha-1}{\alpha}}} = -\alpha c\frac{1+rz}{\alpha\sigma+rz} = -\alpha c\frac{K+r^{\alpha-1}}{K+\alpha\sigma r^{\alpha-1}},\\
\frac{u_{12}}{u_{11}} &= -\frac{(1-\rho)(1-\alpha\sigma)\gamma^{\frac{\alpha-1}{\alpha}}z^{-\frac{1}{\alpha}}}{\alpha\sigma\rho+(1-\rho)(\gamma z)^{\frac{\alpha-1}{\alpha}}} = -\frac{K^{\frac{1}{\alpha}}(1-\alpha\sigma)z^{-\frac{1}{\alpha}}}{\alpha\sigma+K^{\frac{1}{\alpha}}z^{\frac{\alpha-1}{\alpha}}} = -\frac{(1-\alpha\sigma)r}{\alpha\sigma+rz} = -\frac{(1-\alpha\sigma)r^{\alpha}}{K+\alpha\sigma r^{\alpha-1}},\\
\frac{u_1+mu_{12}}{u_{11}} &= -\frac{\alpha c\left(K+r^{\alpha-1}\right)+(1-\alpha\sigma)mr^{\alpha}}{K+\alpha\sigma r^{\alpha-1}} = -c\frac{\alpha\left(K+r^{\alpha-1}\right)+K(1-\alpha\sigma)}{K+\alpha\sigma r^{\alpha-1}}\\
&= -c\frac{(1+\alpha-\alpha\sigma)K+\alpha r^{\alpha-1}}{K+\alpha\sigma r^{\alpha-1}},\\
m\left(J_1\frac{u_{12}}{u_{11}}-J_2\right) &= mJ_1\left(\frac{u_{12}}{u_{11}}+\frac{1}{z}\right) = \frac{(1-\rho)\gamma^{\frac{\alpha-1}{\alpha}}}{\rho\alpha}z^{1-\frac{1}{\alpha}}\left(-\frac{(1-\alpha\sigma)r^{\alpha}}{K+\alpha\sigma r^{\alpha-1}}+\frac{r^{\alpha}}{K}\right)\\
&= \frac{K^{\frac{1}{\alpha}}z^{-\frac{1}{\alpha}}}{\alpha}z\frac{r^{\alpha}}{K}\left(1-\frac{K(1-\alpha\sigma)}{K+\alpha\sigma r^{\alpha-1}}\right) = \frac{r}{\alpha}\frac{K+\alpha\sigma r^{\alpha-1}-K+K\alpha\sigma}{K+\alpha\sigma r^{\alpha-1}} = \frac{\sigma\left(Kr+r^{\alpha}\right)}{K+\alpha\sigma r^{\alpha-1}},\\
mJ_2\frac{u_1}{u_{11}} &= -m\frac{(1-\rho)\gamma^{\frac{\alpha-1}{\alpha}}}{\rho\alpha}\frac{z^{-1-\frac{1}{\alpha}}}{c}\left(-\alpha c\frac{K+r^{\alpha-1}}{K+\alpha\sigma r^{\alpha-1}}\right) = K^{\frac{1}{\alpha}}z^{-\frac{1}{\alpha}}c\frac{K+r^{\alpha-1}}{K+\alpha\sigma r^{\alpha-1}} = c\frac{Kr+r^{\alpha}}{K+\alpha\sigma r^{\alpha-1}}.
\end{aligned}$$

Plugging these expressions into (9) yields

$$\begin{aligned}
&\left(\lambda^2-\delta\lambda\right)\left(-\lambda+\frac{\sigma\left(Kr+r^{\alpha}\right)}{K+\alpha\sigma r^{\alpha-1}}\right)+f''\left(-c\frac{(1+\alpha-\alpha\sigma)K+\alpha r^{\alpha-1}}{K+\alpha\sigma r^{\alpha-1}}\lambda+c\frac{Kr+r^{\alpha}}{K+\alpha\sigma r^{\alpha-1}}\right)\\
=\;&\sigma\left(\lambda^2-\delta\lambda\right)\left(-\frac{\lambda}{\sigma}+\frac{Kr+r^{\alpha}}{K+\alpha\sigma r^{\alpha-1}}\right)+f''c\frac{Kr+r^{\alpha}}{K+\alpha\sigma r^{\alpha-1}}-f''c\frac{(1+\alpha-\alpha\sigma)K+\alpha r^{\alpha-1}}{K+\alpha\sigma r^{\alpha-1}}\lambda\\
=\;&\left(\sigma\left(\lambda^2-\delta\lambda\right)+f''c\right)\left(-\frac{\lambda}{\sigma}+\frac{Kr+r^{\alpha}}{K+\alpha\sigma r^{\alpha-1}}\right)+f''c\frac{\lambda}{\sigma}-f''c\frac{(1+\alpha-\alpha\sigma)K+\alpha r^{\alpha-1}}{K+\alpha\sigma r^{\alpha-1}}\lambda\\
=\;&\left(\frac{Kr+r^{\alpha}}{K+\alpha\sigma r^{\alpha-1}}-\frac{\lambda}{\sigma}\right)\left(\sigma\left(\lambda^2-\delta\lambda\right)+f''c\right)+f''c\frac{\frac{1}{\sigma}\left(K+\alpha\sigma r^{\alpha-1}\right)-(1+\alpha-\alpha\sigma)K-\alpha r^{\alpha-1}}{K+\alpha\sigma r^{\alpha-1}}\lambda\\
=\;&\left(\frac{Kr+r^{\alpha}}{K+\alpha\sigma r^{\alpha-1}}-\frac{\lambda}{\sigma}\right)\left(\sigma\left(\lambda^2-\delta\lambda\right)+f''c\right)+f''c\frac{K}{\sigma}\frac{(1-\sigma)(1-\alpha\sigma)}{K+\alpha\sigma r^{\alpha-1}}\lambda = \frac{\Psi(\lambda,\theta)}{K+\alpha\sigma r^{\alpha-1}},
\end{aligned}$$

if Ψ is defined as in (18). From this first form of Ψ, it is a trivial task to obtain the second, (19):

$$\begin{aligned}
&\left(Kr+r^{\alpha}-\frac{K+\alpha\sigma r^{\alpha-1}}{\sigma}\lambda\right)\left(\sigma\left(\lambda^2-\delta\lambda\right)+f''c\right)+f''c\frac{K(1-\sigma)(1-\alpha\sigma)}{\sigma}\lambda\\
=\;&\left(Kr+r^{\alpha}-\alpha r^{\alpha-1}\lambda-\frac{K}{\sigma}\lambda\right)\sigma\left(\lambda^2-\delta\lambda\right)+f''c\left(Kr+r^{\alpha}-\alpha r^{\alpha-1}\lambda-\frac{K}{\sigma}\lambda\right)+f''c\frac{K\left(1-\sigma-\alpha\sigma+\alpha\sigma^2\right)}{\sigma}\lambda\\
=\;&\left(Kr+r^{\alpha}-\alpha r^{\alpha-1}\lambda\right)\left(\lambda^2-\delta\lambda\right)\sigma-K\lambda\left(\lambda^2-\delta\lambda\right)+f''c\left(Kr+r^{\alpha}-\alpha r^{\alpha-1}\lambda\right)+f''cK(-1-\alpha+\alpha\sigma)\lambda\\
=\;&\left(\left(Kr+r^{\alpha}-\alpha r^{\alpha-1}\lambda\right)\left(\lambda^2-\delta\lambda\right)+f''cK\alpha\lambda\right)\sigma-K\lambda\left(\lambda^2-\delta\lambda\right)+f''c\left(Kr+r^{\alpha}-\alpha r^{\alpha-1}\lambda-K(1+\alpha)\lambda\right).
\end{aligned}$$

Appendix E

Here we are concerned with the side limits of $s\left(\bar{\lambda}\right)$ when $\alpha\downarrow 0$ and when $\alpha\uparrow 1$. From (22), we immediately get $\lim_{\alpha\to 0_+}\bar{\lambda}=-\infty$ and $\lim_{\alpha\to 1_-}\bar{\lambda}=-\infty$. It follows that

$$\begin{aligned}
\lim\quad&\frac{Kr+r^{\alpha}-\alpha r^{\alpha-1}\bar{\lambda}-K(1+\alpha)\bar{\lambda}}{\left(Kr+r^{\alpha}-\alpha r^{\alpha-1}\bar{\lambda}\right)\left(\bar{\lambda}^2-\delta\bar{\lambda}\right)+f''(k^*)c^*K\alpha\bar{\lambda}} = \lim\frac{Kr+r^{\alpha}-\alpha r^{\alpha-1}\bar{\lambda}}{\left(Kr+r^{\alpha}-\alpha r^{\alpha-1}\bar{\lambda}\right)\left(\bar{\lambda}^2-\delta\bar{\lambda}\right)+f''(k^*)c^*K\alpha\bar{\lambda}}-\\
&\lim\frac{K(1+\alpha)}{\left(Kr+r^{\alpha}-\alpha r^{\alpha-1}\bar{\lambda}\right)\left(\bar{\lambda}-\delta\right)+f''(k^*)c^*K\alpha} = 0-0=0,
\end{aligned}$$

no matter which side limit is being considered. Additionally, from (5) and (6) we know that neither k^* nor c^* depend on α. We thus have, from (23) and (22),

$$\begin{aligned}
\lim s\left(\bar{\lambda}\right) &= \lim \frac{K\bar{\lambda}\left(\bar{\lambda}^2 - \delta\bar{\lambda}\right)}{\left(Kr + r^\alpha - \alpha r^{\alpha-1}\bar{\lambda}\right)\left(\bar{\lambda}^2 - \delta\bar{\lambda}\right) + f''\left(k^*\right)c^* K\alpha\bar{\lambda}} \\
&= \lim \frac{K\bar{\lambda}\left(\bar{\lambda} - \delta\right)}{\left(Kr + r^\alpha - \alpha r^{\alpha-1}\bar{\lambda}\right)\left(\bar{\lambda} - \delta\right) + f''\left(k^*\right)c^* K\alpha} = K \lim \frac{\bar{\lambda}}{Kr + r^\alpha - \alpha r^{\alpha-1}\bar{\lambda}} \\
&= -K \lim \frac{\frac{K + \alpha r^{\alpha-1}}{\alpha(1-\alpha)r^{\alpha-2}}}{Kr + r^\alpha + \frac{Kr + \alpha r^\alpha}{1-\alpha}} = -K \lim \frac{K + \alpha r^{\alpha-1}}{\alpha r^{\alpha-1}\left(\left(2-\alpha\right)K + r^{\alpha-1}\right)},
\end{aligned}$$

so that

$$\begin{aligned}
\lim_{\alpha\to 0_+} s\left(\bar{\lambda}\right) &= -K \lim_{\alpha\to 0_+} \frac{K}{\alpha r^{-1}\left(2K + r^{-1}\right)} = -\infty, \\
\lim_{\alpha\to 1_-} s\left(\bar{\lambda}\right) &= -K.
\end{aligned}$$